HL Paper 2

A particle can move along a straight line from a point O. The velocity v, in ms^{-1}, is given by the function $v(t) = 1 - \mathrm{e}^{-\sin t^2}$ where time $t \geq 0$ is measured in seconds.

a. Write down the first two times $t_1,\ t_2 > 0$, when the particle changes direction. [2]

b. (i) Find the time $t < t_2$ when the particle has a maximum velocity. [4]

(ii) Find the time $t < t_2$ when the particle has a minimum velocity.

c. Find the distance travelled by the particle between times $t = t_1$ and $t = t_2$. [2]

Markscheme

a. $t_1 = 1.77\ (\mathrm{s}) \quad \left(= \sqrt{\pi}\ (\mathrm{s})\right) \quad \text{and} \quad t_2 = 2.51\ (\mathrm{s}) \quad \left(= \sqrt{2\pi}\ (\mathrm{s})\right)$ ***A1A1***

[2 marks]

b. (i) attempting to find (graphically or analytically) the first $t_{\max}$ ***(M1)***

$t = 1.25\ (\mathrm{s}) \quad \left(= \sqrt{\frac{\pi}{2}}\ (\mathrm{s})\right)$ ***A1***

(ii) attempting to find (graphically or analytically) the first $t_{\min}$ ***(M1)***

$t = 2.17\ (\mathrm{s}) \quad \left(= \sqrt{\frac{3\pi}{2}}\ (\mathrm{s})\right)$ ***A1***

[4 marks]

c. distance travelled $= \left|\int_{1.772\ldots}^{2.506\ldots} 1 - \mathrm{e}^{-\sin t^2}\mathrm{d}t\right|$ (or equivalent) ***(M1)***

$= 0.711\ (\mathrm{m})$ ***A1***

Note: Award ***M1*** for attempting to form a definite integral involving $1 - \mathrm{e}^{-\sin t^2}$. To award the ***A1***, correct limits leading to 0.711 must include the use of absolute value or a statement such as “distance must be positive”.

In part (c), award ***A1FT*** for a candidate working in degree mode $(5.39\ (\mathrm{m}))$.

[2 marks]

Total [8 marks]

Examiners report

a. [N/A]

b. [N/A]

c. [N/A]

Consider the function $f(x) = x^3 - 3x^2 - 9x + 10$, $x \in \mathbb{R}$.

a. Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$. [4]

b. Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line. [2]

Markscheme

a. $f'(x) = 3x^2 - 6x - 9\ (= 0)$ ***(M1)***

$(x+1)(x-3) = 0$

$x = -1$; $x = 3$

(max) (−1, 15); (min) (3, −17) ***A1A1***

Note: The coordinates need not be explicitly stated but the values need to be seen.

$y = -8x + 7$ ***A1*** ***N2***

[4 marks]

b. $f''(x) = 6x - 6 = 0 \Rightarrow$ inflexion (1, −1) ***A1***

which lies on $y = -8x + 7$ ***R1AG***

[2 marks]

Examiners report

a. There were a significant number of completely correct answers to this question. Many candidates demonstrated a good understanding of basic differential calculus in the context of coordinate geometry whilst other used technology to find the turning points.

b. There were a significant number of completely correct answers to this question. Many candidates demonstrated a good understanding of basic differential calculus in the context of coordinate geometry whilst other used technology to find the turning points. There were many correct demonstrations of the “show that” in (b).

Let $f(x) = x^4 + 0.2x^3 - 5.8x^2 - x + 4,\ x \in \mathbb{R}$.

The domain of f is now restricted to $[0,\ a]$.

Let $g(x) = 2\sin(x-1) - 3,\ \ -\frac{\pi}{2} + 1 \leqslant x \leqslant \frac{\pi}{2} + 1$.

a. Find the solutions of $f(x) > 0$. [3]

b. For the curve $y = f(x)$. [5]

(i) Find the coordinates of both local minimum points.

(ii) Find the x-coordinates of the points of inflexion.

c.i. Write down the largest value of a for which f has an inverse. Give your answer correct to 3 significant figures. [2]

c.ii. For this value of *a* sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, showing clearly the coordinates of the end points of each curve. [2]

c.iii. Solve $f^{-1}(x) = 1$. [2]

d.i. Find an expression for $g^{-1}(x)$, stating the domain. [4]

d.ii. Solve $(f^{-1} \circ g)(x) < 1$. [4]

Markscheme

a. valid method *eg*, sketch of curve or critical values found ***(M1)***

$x < -2.24,\ x > 2.24,$ ***A1***

$-1 < x < 0.8$ ***A1***

Note: Award ***M1A1A0*** for correct intervals but with inclusive inequalities.

[3 marks]

b. (i) $(1.67,\ -5.14),\ (-1.74,\ -3.71)$ ***A1A1***

Note: Award ***A1A0*** for any two correct terms.

(ii) $f'(x) = 4x^3 + 0.6x^2 - 11.6x - 1$

$f''(x) = 12x^2 + 1.2x - 11.6 = 0$ ***(M1)***

$-1.03,\ 0.934$ ***A1A1***

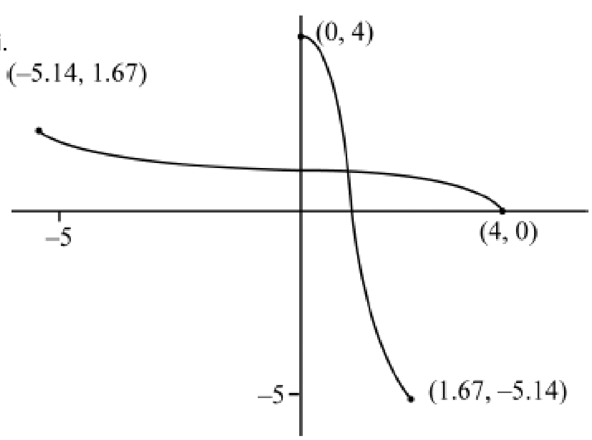
Note: ***M1*** should be awarded if graphical method to find zeros of $f''(x)$ or turning points of $f'(x)$ is shown.

[5 marks]

c.i. 1.67 ***A1***

[2 marks]

c.ii.



M1A1A1

Note: Award ***M1*** for reflection of their $y = f(x)$ in the line $y = x$ provided their f is one-one.

A1 for $(0,\ 4)$, $(4,\ 0)$ (Accept axis intercept values) ***A1*** for the other two sets of coordinates of other end points

[2 marks]

c.iii. $x = f(1)$ ***M1***

$= -1.6$ ***A1***

[2 marks]

d.i. $y = 2\sin(x-1) - 3$

$x = 2\sin(y-1) - 3$ ***(M1)***

$\left(g^{-1}(x) =\right)\ \arcsin\left(\frac{x+3}{2}\right) + 1$ ***A1***

$-5 \leqslant x \leqslant -1$ ***A1A1***

Note: Award ***A1*** for –5 and –1, and ***A1*** for correct inequalities if numbers are reasonable.

[8 marks]

d.ii. $f^{-1}\left(g(x)\right) < 1$

$g(x) > -1.6$ ***(M1)***

$x > g^{-1}(-1.6) = 1.78$ ***(A1)***

Note: Accept = in the above.

$1.78 < x \leqslant \frac{\pi}{2} + 1$ ***A1A1***

Note: ***A1*** for $x > 1.78$ (allow ≥) and ***A1*** for $x \leqslant \frac{\pi}{2} + 1$.

[4 marks]

Examiners report

a. Parts (a) and (b) were well answered, with considerably less success in part (c). Surprisingly few students were able to reflect the curve in $y = x$ satisfactorily, and many were not making their sketch using the correct domain.

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d.i. Part d(i) was generally well done, but there were few correct answers for d(ii).

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a. Given that $2x^3 - 3x + 1$ can be expressed in the form $Ax\left(x^2+1\right) + Bx + C$, find the values of the constants A, B and C. [2]

b. Hence find $\int \frac{2x^3-3x+1}{x^2+1}\mathrm{d}x$. [5]

Markscheme

a. $2x^3 - 3x + 1 = Ax\left(x^2+1\right) + Bx + C$

$A = 2,\ C = 1,$ ***A1***

$A + B = -3 \Rightarrow B = -5$ ***A1***

[2 marks]

b. $\int \frac{2x^3-3x+1}{x^2+1}\mathrm{d}x = \int \left(2x - \frac{5x}{x^2+1} + \frac{1}{x^2+1}\right)\mathrm{d}x$ ***M1M1***

Note: Award ***M1*** for dividing by $\left(x^2+1\right)$ to get $2x$, ***M1*** for separating the $5x$ and 1.

$= x^2 - \frac{5}{2}\ln\left(x^2+1\right) + \arctan x\ (+c)$ ***(M1)A1A1***

Note: Award ***(M1)A1*** for integrating $\frac{5x}{x^2+1}$, ***A1*** for the other two terms.

[5 marks]

Examiners report

a. [N/A]

b. [N/A]

The graphs of $y = x^2\mathrm{e}^{-x}$ and $y = 1 - 2\sin x$ for $2 \leqslant x \leqslant 7$ intersect at points A and B.

The *x*-coordinates of A and B are x_{A} and x_{B}.

a. Find the value of x_{A} and the value of x_{B}. [2]

b. Find the area enclosed between the two graphs for $x_{\mathbf{A}} \leqslant x \leqslant x_{\mathrm{B}}$. [3]

Markscheme

a. $x_{\mathrm{A}} = 2.87$ ***A1***

$x_{\mathrm{B}} = 6.78$ ***A1***

[2 marks]

b. $\int_{2.87172\mathrm{K}}^{6.77681K} 1 - 2\sin x - x^2\mathrm{e}^{-x}\mathrm{d}x$ ***(M1)(A1)***

$= 6.76$ ***A1***

Note: Award ***(M1)*** for definite integral and ***(A1)*** for a correct definite integral.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

A particle moves in a straight line, its velocity $v\ \mathrm{ms}^{-1}$ at time t seconds is given by $v = 9t - 3t^2,\ 0 \le t \le 5$.

At time $t = 0$, the displacement s of the particle from an origin O is 3 m.

a. Find the displacement of the particle when $t = 4$. [3]

b. Sketch a displacement/time graph for the particle, $0 \le t \le 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. [5]

c. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \ge 0$. [3]

Given further that $s = 16.5$ when $t = 7.5$, find the values of a and b.

d. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ such that s is continuous for all $t \ge 0$. [4]

Find the times t_1 and $t_2 (0 < t_1 < t_2 < 8)$ when the particle returns to its starting point.

Markscheme

a. **METHOD 1**

$s = \int (9t - 3t^2)\mathrm{d}t = \frac{9}{2}t^2 - t^3 (+c)$ ***(M1)***

$t = 0,\ s = 3 \Rightarrow c = 3$ ***(A1)***

$t = 4 \Rightarrow s = 11$ ***A1***

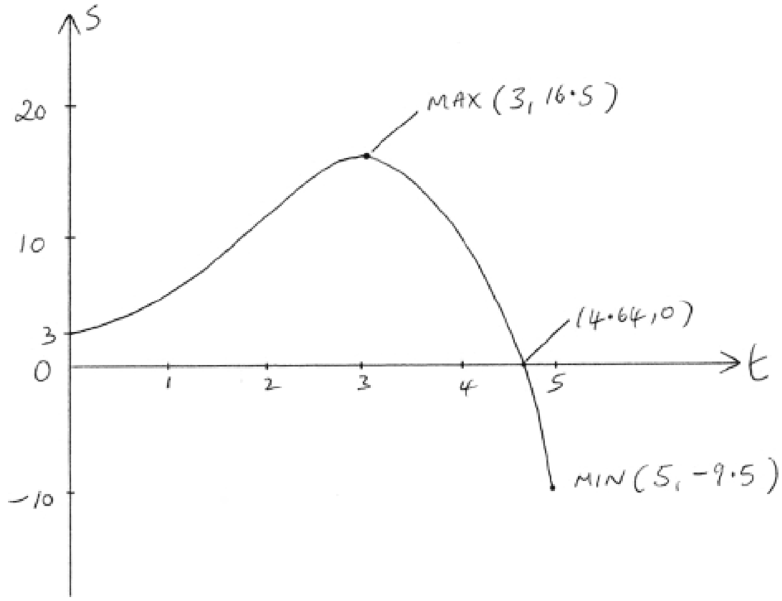
METHOD 2

$s = 3 + \int_0^4 (9t - 3t^2)\mathrm{d}t$ ***(M1)(A1)***

$s = 11$ ***A1***

[3 marks]

b.



correct shape over correct domain ***A1***

maximum at $(3,\ 16.5)$ ***A1***

t intercept at 4.64, s intercept at 3 ***A1***

minimum at $(5,\ -9.5)$ ***A1***

[5 marks]

c. $-9.5 = a + b\cos 2\pi$

$16.5 = a + b\cos 3\pi$ ***(M1)***

Note: Only award ***M1*** if two simultaneous equations are formed over the correct domain.

$a = \frac{7}{2}$ ***A1***

$b = -13$ ***A1***

[3 marks]

d. at t_1:

$3 + \frac{9}{2}t^2 - t^3 = 3$ ***(M1)***

$t^2\left(\frac{9}{2} - t\right) = 0$

$t_1 = \frac{9}{2}$ ***A1***

solving $\frac{7}{2} - 13\cos\frac{2\pi t}{5} = 3$ ***(M1)***

$\text{GDC} \Rightarrow t_2 = 6.22$ ***A1***

Note: Accept graphical approaches.

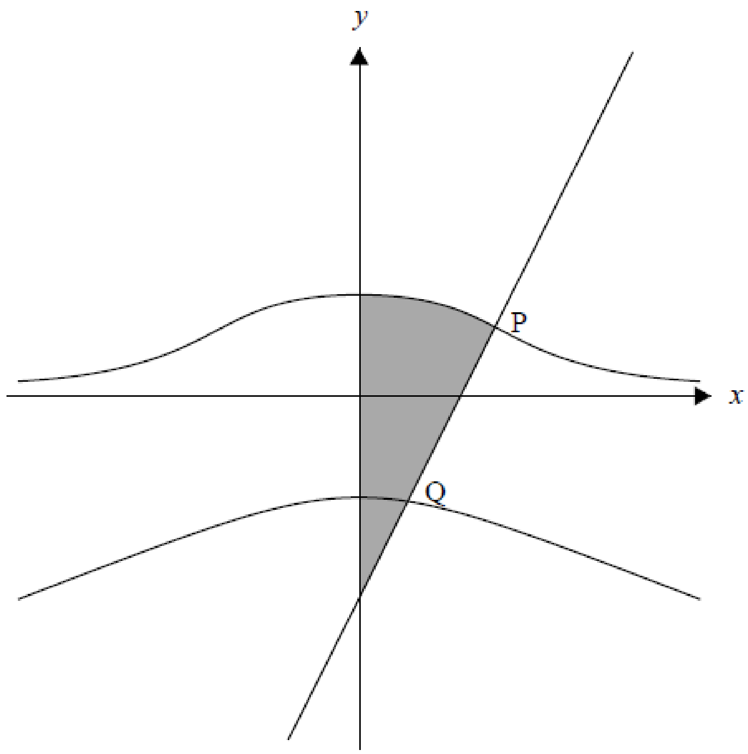
[4 marks]

Total [15 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The following graph shows the two parts of the curve defined by the equation $x^2y = 5 - y^4$, and the normal to the curve at the point P(2 , 1).



a. Show that there are exactly two points on the curve where the gradient is zero. [7]

b. Find the equation of the normal to the curve at the point P. [5]

c. The normal at P cuts the curve again at the point Q. Find the x-coordinate of Q. [3]

d. The shaded region is rotated by 2π about the y-axis. Find the volume of the solid formed. [7]

Markscheme

a. differentiating implicitly: ***M1***

$2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = -4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

Note: Award ***A1*** for each side.

if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ then either $x = 0$ or $y = 0$ ***M1A1***

$x = 0 \Rightarrow$ two solutions for y $\left(y = \pm\sqrt[4]{5}\right)$ ***R1***

$y = 0$ not possible (as 0 ≠ 5) ***R1***

hence exactly two points ***AG***

Note: For a solution that only refers to the graph giving two solutions at $x = 0$ and no solutions for $y = 0$ award ***R1*** only.

[7 marks]

b. at (2, 1) $4+4\frac{\mathrm{d}y}{\mathrm{d}x}=-4\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}$ ***(A1)***

gradient of normal is 2 ***M1***

1 = 4 + c ***(M1)***

equation of normal is $y=2x-3$ ***A1***

[5 marks]

c. substituting ***(M1)***

$x^2(2x-3)=5-(2x-3)^4$ or $\left(\frac{y+3}{2}\right)^2 y=5-y^4$ ***(A1)***

$x=0.724$ ***A1***

[3 marks]

d. recognition of two volumes ***(M1)***

volume $1=\pi\int_1^{\sqrt[4]{5}}\frac{5-y^4}{y}\mathrm{d}y\,(=101\pi=3.178\ldots)$ ***M1A1A1***

Note: Award ***M1*** for attempt to use $\pi\int x^2\mathrm{d}y$, ***A1*** for limits, ***A1*** for $\frac{5-y^4}{y}$ Condone omission of π at this stage.

volume 2

EITHER

$=\frac{1}{3}\pi\times 2^2\times 4\,(=16.75\ldots)$ ***(M1)(A1)***

OR

$=\pi\int_{-3}^{1}\left(\frac{y+3}{2}\right)^2\mathrm{d}y\left(=\frac{16\pi}{3}=16.75\ldots\right)$ ***(M1)(A1)***

THEN

total volume = 19.9 ***A1***

[7 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

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a. Find $\int x\sec^2 x\mathrm{d}x$. [4]

b. Determine the value of m if $\int_0^m x\sec^2 x\mathrm{d}x=0.5$, where $m>0$. [2]

Markscheme

a. $\int x\sec^2 x\mathrm{d}x=x\tan x-\int 1\times\tan x\mathrm{d}x$ ***M1A1***

$=x\tan x+\ln|\cos x|(+c)\ \ (=x\tan x-\ln|\sec x|(+c))$ ***M1A1***

[4 marks]

b. attempting to solve an appropriate equation *eg* $m\tan m + \ln(\cos m) = 0.5$ ***(M1)***

$m = 0.822$ ***A1***

Note: Award ***A1*** if $m = 0.822$ is specified with other positive solutions.

[2 marks]

Examiners report

a. In part (a), a large number of candidates were able to use integration by parts correctly but were unable to use integration by substitution to then find the indefinite integral of tan x. In part (b), a large number of candidates attempted to solve the equation without direct use of a GDC's numerical solve command. Some candidates stated more than one solution for m and some specified m correct to two significant figures only.

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The curve $y = \mathrm{e}^{-x} - x + 1$ intersects the x-axis at P.

(a) Find the x-coordinate of P.

(b) Find the area of the region completely enclosed by the curve and the coordinate axes.

Markscheme

(a) Either solving $\mathrm{e}^{-x} - x + 1 = 0$ for x, stating $\mathrm{e}^{-x} - x + 1 = 0$, stating P($x$, 0) or using an appropriate sketch graph. ***M1***

$x = 1.28$ ***A1*** ***N1***

Note: Accept P(1.28, 0) .

(b) Area $= \int_0^{1.278...} (\mathrm{e}^{-x} - x + 1)\mathrm{d}x$ ***M1A1***

$= 1.18$ ***A1*** ***N1***

Note: Award ***M1A0A1*** if the dx is absent.

[5 marks]

Examiners report

This was generally well done. In part (a), most candidates were able to find $x = 1.28$ successfully. A significant number of candidates were awarded an accuracy penalty for expressing answers to an incorrect number of significant figures.

Part (b) was generally well done. A number of candidates unfortunately omitted the dx in the integral while some candidates omitted to write down the definite integral and instead offered detailed instructions on how they obtained the answer using their GDC.

The functions f and g are defined by

$$f(x) = \frac{e^x + e^{-x}}{2},\ x \in \mathbb{R}$$

$$g(x) = \frac{e^x - e^{-x}}{2},\ x \in \mathbb{R}$$

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R},\ n > 1$.

Let $t(x) = \frac{g(x)}{f(x)}$.

a. (i) Show that $\frac{1}{4f(x)-2g(x)} = \frac{e^x}{e^{2x}+3}$. [9]

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x)-2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a,\ b \in \mathbb{Z}^+$.

b. (i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$. [8]

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$.

c. (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$. [6]

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$.

Markscheme

a. (i) $\frac{1}{4\left(\frac{e^x+e^{-x}}{2}\right)-2\left(\frac{e^x-e^{-x}}{2}\right)}$ ***(M1)***

$= \frac{1}{2(e^x+e^{-x})-(e^x-e^{-x})}$ ***(A1)***

$= \frac{1}{e^x+3e^{-x}}$ ***A1***

$= \frac{e^x}{e^{2x}+3}$ ***AG***

(ii) $u = e^x \Rightarrow du = e^x dx$ ***A1***

$\int \frac{e^x}{e^{2x}+3} dx = \int \frac{1}{u^2+3} du$ ***M1***

(when $x = 0,\ u = 1$ and when $x = \ln 3,\ u = 3$)

$\int_1^3 \frac{1}{u^2+3} du \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\right]_1^3$ ***M1A1***

$\left(= \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{e^x}{\sqrt{3}}\right)\right]_0^{\ln 3}\right)$

$= \frac{\pi\sqrt{3}}{9} - \frac{\pi\sqrt{3}}{18}$ ***(M1)***

$= \frac{\pi\sqrt{3}}{18}$ ***A1***

[9 marks]

b. (i) $(n+1)\mathrm{e}^{2x} - 2k\mathrm{e}^{x} + (n-1) = 0$ ***M1A1***

$\mathrm{e}^{x} = \frac{2k \pm \sqrt{4k^2 - 4(n^2-1)}}{2(n+1)}$ ***M1***

$x = \ln\left(\frac{k \pm \sqrt{k^2 - n^2 + 1}}{n+1}\right)$ ***M1A1***

(ii) for two real solutions, we require $k > \sqrt{k^2 - n^2 + 1}$ ***R1***

and we also require $k^2 - n^2 + 1 > 0$ ***R1***

$k^2 > n^2 - 1$ ***A1***

$\Rightarrow k > \sqrt{n^2 - 1}$ ($k \in \mathbb{R}^{+}$) ***AG***

[8 marks]

c. **METHOD 1**

$t(x) = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{\mathrm{e}^{x} + \mathrm{e}^{-x}}$

$t'(x) = \frac{(\mathrm{e}^{x} + \mathrm{e}^{-x})^2 - (\mathrm{e}^{x} - \mathrm{e}^{-x})^2}{(\mathrm{e}^{x} + \mathrm{e}^{-x})^2}$ ***M1A1***

$t'(x) = \frac{\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}\right)^2 - \left(\frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}\right)^2}{\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}\right)^2}$ ***A1***

$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

METHOD 2

$t'(x) = \frac{f(x)g'(x) = g(x)f'(x)}{[f(x)]^2}$ ***M1A1***

$g'(x) = f(x)$ and $f'(x) = g(x)$ ***A1***

$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

METHOD 3

$t(x) = (\mathrm{e}^{x} - \mathrm{e}^{-x})(\mathrm{e}^{x} + \mathrm{e}^{-x})^{-1}$

$t'(x) = 1 - \frac{(\mathrm{e}^{x} - \mathrm{e}^{-x})^2}{(\mathrm{e}^{x} + \mathrm{e}^{-x})^2}$ ***M1A1***

$= 1 - \frac{[g(x)]^2}{[f(x)]^2}$ ***A1***

$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

METHOD 4

$t'(x) = \frac{g'(x)}{f(x)} - \frac{g(x)f'(x)}{[f(x)]^2}$ ***M1A1***

$g'(x) = f(x)$ and $f'(x) = g(x)$ gives $t'(x) = 1 - \frac{[g(x)]^2}{[f(x)]^2}$ ***A1***

$= \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ ***AG***

(ii) **METHOD 1**

$[f(x)]^2 > [g(x)]^2$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0,\ x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}$ and $g(x) = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}$ or $\mathrm{e}^{x} + \mathrm{e}^{-x}$ and $\mathrm{e}^{x} - \mathrm{e}^{-x}$.

METHOD 2

$[f(x)]^2 - [g(x)]^2 = 1$ (or equivalent) ***M1A1***

$[f(x)]^2 > 0$ ***R1***

hence $t'(x) > 0,\ x \in \mathbb{R}$ ***AG***

Note: Award as above for use of either $f(x) = \frac{e^x + e^{-x}}{2}$ and $g(x) = \frac{e^x - e^{-x}}{2}$ or $e^x + e^{-x}$ and $e^x - e^{-x}$.

METHOD 3

$t'(x) = \frac{4}{(e^x+e^{-x})^2}$

$(e^x + e^{-x})^2 > 0$ ***M1A1***

$\frac{4}{(e^x+e^{-x})^2} > 0$ ***R1***

hence $t'(x) > 0,\ x \in \mathbb{R}$ ***AG***

[6 marks]

Examiners report

a. Parts (a) and (c) were accessible to the large majority of candidates. Candidates found part (b) considerably more challenging.

Part (a)(i) was reasonably well done with most candidates able to show that $\frac{1}{4f(x)-2g(x)} = \frac{e^x}{e^{2x}+3}$. In part (a)(ii), a number of candidates correctly used the required substitution to obtain $\int \frac{e^x}{e^{2x}+3} dx = \int \frac{1}{u^2+3} du$ but then thought that the antiderivative involved natural log rather than arctan.

b. Parts (a) and (c) were accessible to the large majority of candidates. Candidates found part (b) considerably more challenging.

In part (b)(i), a reasonable number of candidates were able to form a quadratic in e^x (involving parameters n and k) and then make some progress towards solving for e^x in terms of n and k. Having got that far, a small number of candidates recognised to then take the natural logarithm of both sides and hence solve $h(x) = k$ for χ. In part (b)(ii), a small number of candidates were able to show from their solutions to part (b)(i) or through the use of the discriminant that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{k^2 - n^2 + 1}$ and $k > \sqrt{n^2 - 1}$.

c. Parts (a) and (c) were accessible to the large majority of candidates. Candidates found part (b) considerably more challenging.

It was pleasing to see the number of candidates who attempted part (c). In part (c)(i), a large number of candidates were able to correctly apply either the quotient rule or the product rule to find $t'(x)$. A smaller number of candidates were then able to show equivalence between the form of $t'(x)$ they had obtained and the form of $t'(x)$ required in the question. A pleasing number of candidates were able to exploit the property that $f'(x) = g(x)$ and $g'(x) = f(x)$. As with part (c)(i), part (c)(ii) could be successfully tackled in a number of ways. The best candidates offered concise logical reasoning to show that $t'(x) > 0$ for $x \in \mathbb{R}$.

Find the gradient of the tangent to the curve $x^3y^2 = \cos(\pi y)$ at the point $(-1, 1)$.

Markscheme

METHOD 1

$3x^2y^2 + 2x^3y\frac{dy}{dx} = -\pi\sin(\pi y)\frac{dy}{dx}$ ***A1A1A1***

At $(-1,\ 1),\ 3 - 2\frac{dy}{dx} = 0$ ***M1A1***

$\frac{dy}{dx} = \frac{3}{2}$ ***A1***

[6 marks]

METHOD 2

$3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} = -\pi\sin(\pi y)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2y^2}{-\pi\sin(\pi y) - 2x^3y}$ ***A1***

At $(-1,\ 1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(-1)^2(1)^2}{-\pi\sin(\pi) - 2(-1)^3(1)} = \frac{3}{2}$ ***M1A1***

[6 marks]

Examiners report

A large number of candidates obtained full marks on this question. Some candidates missed π and/or $\frac{\mathrm{d}y}{\mathrm{d}x}$ when differentiating the trigonometric function. Some candidates attempted to rearrange before differentiating, and some made algebraic errors in rearranging.

Consider the function $f(x) = 2\sin^2 x + 7\sin 2x + \tan x - 9,\ 0 \leqslant x < \frac{\pi}{2}$.

Let $u = \tan x$.

a.i. Determine an expression for $f'(x)$ in terms of x. [2]

a.ii. Sketch a graph of $y = f'(x)$ for $0 \leqslant x < \frac{\pi}{2}$. [4]

a.iii. Find the x-coordinate(s) of the point(s) of inflexion of the graph of $y = f(x)$, labelling these clearly on the graph of $y = f'(x)$. [2]

b.i. Express $\sin x$ in terms of μ. [2]

b.ii. Express $\sin 2x$ in terms of u. [3]

b.iii. Hence show that $f(x) = 0$ can be expressed as $u^3 - 7u^2 + 15u - 9 = 0$. [2]

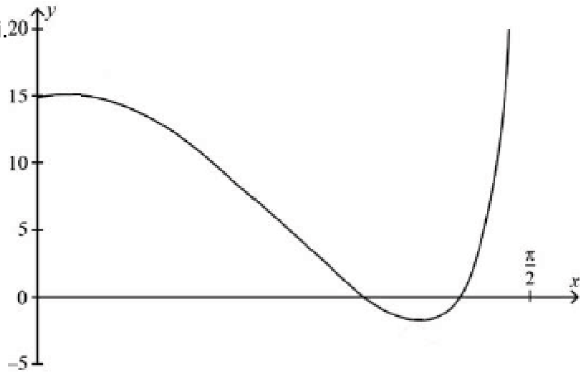
c. Solve the equation $f(x) = 0$, giving your answers in the form $\arctan k$ where $k \in \mathbb{Z}$. [3]

Markscheme

a.i. $f'(x) = 4\sin x\cos x + 14\cos 2x + \sec^2 x$ (or equivalent) ***(M1)A1***

[2 marks]

a.ii.



A1A1A1A1

Note: Award ***A1*** for correct behaviour at $x = 0$, ***A1*** for correct domain and correct behaviour for $x \to \frac{\pi}{2}$, ***A1*** for two clear intersections with x-axis and minimum point, ***A1*** for clear maximum point.

[4 marks]

a.iii. $x = 0.0736$ ***A1***

$x = 1.13$ ***A1***

[2 marks]

b.i. attempt to write $\sin x$ in terms of u only ***(M1)***

$\sin x = \frac{u}{\sqrt{1+u^2}}$ ***A1***

[2 marks]

b.ii. $\cos x = \frac{1}{\sqrt{1+u^2}}$ ***(A1)***

attempt to use $\sin 2x = 2 \sin x \cos x$ $\left(= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}\right)$ ***(M1)***

$\sin 2x = \frac{2u}{1+u^2}$ ***A1***

[3 marks]

b.iii. $2\sin^2 x + 7\sin 2x + \tan x - 9 = 0$

$\frac{2u^2}{1+u^2} + \frac{14u}{1+u^2} + u - 9\ (= 0)$ ***M1***

$\frac{2u^2+14u+u(1+u^2)-9(1+u^2)}{1+u^2} = 0$ (or equivalent) ***A1***

$u^3 - 7u^2 + 15u - 9 = 0$ ***AG***

[2 marks]

c. $u = 1$ or $u = 3$ ***(M1)***

$x = \arctan(1)$ ***A1***

$x = \arctan(3)$ ***A1***

Note: Only accept answers given the required form.

[3 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
a.iii. [N/A]
b.i. [N/A]
b.ii. [N/A]
b.iii. [N/A]
c. [N/A]

A function f is defined by $f(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$, $x \in \mathbb{R}$.

a. (i) Explain why the inverse function f^{-1} does not exist. [14]

(ii) Show that the equation of the normal to the curve at the point P where $x = \ln 3$ is given by $9x + 12y - 9\ln 3 - 20 = 0$.

(iii) Find the x-coordinates of the points Q and R on the curve such that the tangents at Q and R pass through $(0, 0)$.

b. The domain of f is now restricted to $x \geqslant 0$. [8]

(i) Find an expression for $f^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y = f(x)$ and the lines $x = 0$ and $y = 5$ is rotated through an angle of 2π radians about the y-axis.

Markscheme

a. (i) either counterexample or sketch or

recognising that $y = k$ $(k > 1)$ intersects the graph of $y = f(x)$ twice ***M1***

function is not $1 - 1$ (does not obey horizontal line test) ***R1***

so f^{-1} does not exist ***AG***

(ii) $f'(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$ ***(A1)***

$f'(\ln 3) = \frac{4}{3}$ $(= 1.33)$ ***(A1)***

$m = -\frac{3}{4}$ ***M1***

$f(\ln 3) = \frac{5}{3}$ $(= 1.67)$ ***A1***

EITHER

$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$ ***M1***

$4y - \frac{20}{3} = -3x + 3\ln 3$ ***A1***

OR

$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$ ***M1***

$c = \frac{5}{3} + \frac{3}{4}\ln 3$

$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$ ***A1***

$12y = -9x + 20 + 9\ln 3$

THEN

$9x + 12y - 9\ln 3 - 20 = 0$ ***AG***

(iii) The tangent at $(a,\ f(a))$ has equation $y - f(a) = f'(a)(x - a)$. ***(M1)***

$f'(a) = \frac{f(a)}{a}$ (or equivalent) ***(A1)***

$\mathrm{e}^a - \mathrm{e}^{-a} = \frac{\mathrm{e}^a + \mathrm{e}^{-a}}{a}$ (or equivalent) ***A1***

attempting to solve for a ***(M1)***

$a = \pm 1.20$ ***A1A1***

[14 marks]

b. (i) $2y = \mathrm{e}^x + \mathrm{e}^{-x}$

$\mathrm{e}^{2x} - 2y\mathrm{e}^x + 1 = 0$ ***M1A1***

Note: Award ***M1*** for either attempting to rearrange or interchanging x and y.

$\mathrm{e}^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$\mathrm{e}^x = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

Note: Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi \int_1^5 \left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \mathrm{d}y$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to use $V = \pi \int_c^d x^2 \mathrm{d}y$.

$= 37.1 \ \left(\text{units}^3\right)$ ***A1***

[8 marks]

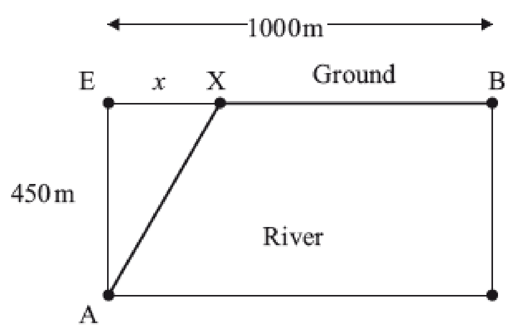
Examiners report

a. In part (a) (i), successful candidates typically sketched the graph of $y = f(x)$, applied the horizontal line test to the graph and concluded that the function was not $1 - 1$ (it did not obey the horizontal line test).
In part (a) (ii), a large number of candidates were able to show that the equation of the normal at point P was $9x + 12y - 9\ln 3 - 20 = 0$. A few candidates used the gradient of the tangent rather than using it to find the gradient of the normal.
Part (a) (iii) challenged most candidates. Most successful candidates graphed $y = f(x)$ and $y = xf'(x)$ on the same set of axes and found the x-coordinates of the intersection points.

b. Part (b) (i) challenged most candidates. While a large number of candidates seemed to understand how to find an inverse function, poor algebra skills (e.g. erroneously taking the natural logarithm of both sides) meant that very few candidates were able to form a quadratic in either e^x or e^y.

Engineers need to lay pipes to connect two cities A and B that are separated by a river of width 450 metres as shown in the following diagram. They plan to lay the pipes under the river from A to X and then under the ground from X to B. The cost of laying the pipes under the river is five times the cost of laying the pipes under the ground.

Let $\text{EX} = x$.



Let k be the cost, in dollars per metre, of laying the pipes under the ground.

(a) Show that the total cost C, in dollars, of laying the pipes from A to B is given by $C = 5k\sqrt{202\,500 + x^2} + (1000 - x)k$.

(b) (i) Find $\frac{\mathrm{d}C}{\mathrm{d}x}$.

(ii) Hence find the value of x for which the total cost is a minimum, justifying that this value is a minimum.

(c) Find the minimum total cost in terms of k.

The angle at which the pipes are joined is $\mathrm{A\hat{X}B} = \theta$.

(d) Find θ for the value of x calculated in (b).

For safety reasons θ must be at least 120°.

Given this new requirement,

(e) (i) find the new value of *x* which minimises the total cost;

(ii) find the percentage increase in the minimum total cost.

Markscheme

(a) $C = \text{AX} \times 5k + \text{XB} \times k$ ***(M1)***

Note: Award ***(M1)*** for attempting to express the cost in terms of AX, XB and *k*.

$= 5k\sqrt{450^2 + x^2} + (1000 - x)k$ ***A1***

$= 5k\sqrt{202\,500 + x^2} + (1000 - x)k$ ***AG***

[2 marks]

(b) (i) $\frac{\mathrm{d}C}{\mathrm{d}x} = k\left[\frac{5\times 2x}{2\sqrt{202\,500+x^2}} - 1\right] = k\left(\frac{5x}{\sqrt{202\,500+x^2}} - 1\right)$ ***M1A1***

Note: Award ***M1*** for an attempt to differentiate and ***A1*** for the correct derivative.

(ii) attempting to solve $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$ ***M1***

$\frac{5}{\sqrt{202\,500+x^2}} = 1$ ***(A1)***

$x = 91.9\ (\mathrm{m})\ \left(= \frac{75\sqrt{6}}{2}\ (\mathrm{m})\right)$ ***A1***

METHOD 1

for example,

at $x = 91\,\frac{\mathrm{d}C}{\mathrm{d}x} = -0.00895k < 0$ ***M1***

at $x = 92\,\frac{\mathrm{d}C}{\mathrm{d}x} = 0.001506k > 0$ ***A1***

Note: Award ***M1*** for attempting to find the gradient either side of $x = 91.9$ and ***A1*** for two correct values.

thus $x = 91.9$ gives a minimum ***AG***

METHOD 2

$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = \frac{1\,012\,500k}{(x^2+202\,500)^{\frac{3}{2}}}$

at $x = 91.9\,\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = 0.010451k > 0$ ***(M1)A1***

Note: Award ***M1*** for attempting to find the second derivative and ***A1*** for the correct value.

Note: If $\frac{\mathrm{d}^2C}{\mathrm{d}x^2}$ is obtained and its value at $x = 91.9$ is not calculated, award ***(M1)A1*** for correct reasoning *eg*, both numerator and denominator are positive at $x = 91.9$.

thus $x = 91.9$ gives a minimum ***AG***

METHOD 3

Sketching the graph of either *C* versus *x* or $\frac{\mathrm{d}C}{\mathrm{d}x}$ versus *x*. ***M1***

Clearly indicating that $x = 91.9$ gives the minimum on their graph. ***A1***

[7 marks]

(c) $C_{\text{min}} = 3205k$ ***A1***

Note: Accept $3200k$.

Accept $3204k$.

[1 mark]

(d) $\arctan\left(\frac{450}{91.855865\text{K}}\right) = 78.463\text{K}^\circ$ ***M1***

$180 - 78.463\text{K} = 101.537\text{K}$

$= 102^\circ$ ***A1***

[2 marks]

(e) (i) when $\theta = 120^\circ,\ x = 260\ (\text{m})\ \left(\frac{450}{\sqrt{3}}\ (\text{m})\right)$ ***A1***

(ii) $\frac{133.728\text{K}}{3204.5407685\text{K}} \times 100\%$ ***M1***

$= 4.17\ (\%\ )$ ***A1***

[3 marks]

Total [15 marks]

Examiners report

[N/A]

The line $y = m(x - m)$ is a tangent to the curve $(1 - x)y = 1$.

Determine *m* and the coordinates of the point where the tangent meets the curve.

Markscheme

EITHER

$y = \frac{1}{1-x} \Rightarrow y' = \frac{1}{(1-x)^2}$ ***M1A1***

solve simultaneously ***M1***

$\frac{1}{1-x} = m(x - m)$ and $\frac{1}{(1-x)^2} = m$

$\frac{1}{1-x} = \frac{1}{(1-x)^2}\left(x - \frac{1}{(1-x)^2}\right)$ ***A1***

Note: Accept equivalent forms.

$(1 - x)^3 - x(1 - x)^2 + 1 = 0,\ x \neq 1$

$x = 1.65729\ldots \Rightarrow y = \frac{1}{1-1.65729\ldots} = -1.521379\ldots$

tangency point (1.66, –1.52) ***A1A1***

$m = (-1.52137\ldots)^2 = 2.31$ ***A1***

OR

$(1 - x)y = 1$

$m(1 - x)(x - m) = 1$ ***M1***

$m(x - x^2 - m + mx) = 1$

$mx^2 - x(m+m^2) + (m^2+1) = 0$ ***A1***

$b^2 - 4ac = 0$ ***(M1)***

$(m+m^2)^2 - 4m(m^2+1) = 0$

$m = 2.31$ ***A1***

substituting $m = 2.31\ldots$ into $mx^2 - x(m+m^2) + (m^2+1) = 0$ ***(M1)***

$x = 1.66$ ***A1***

$y = \frac{1}{1-1.65729} = -1.52$ ***A1***

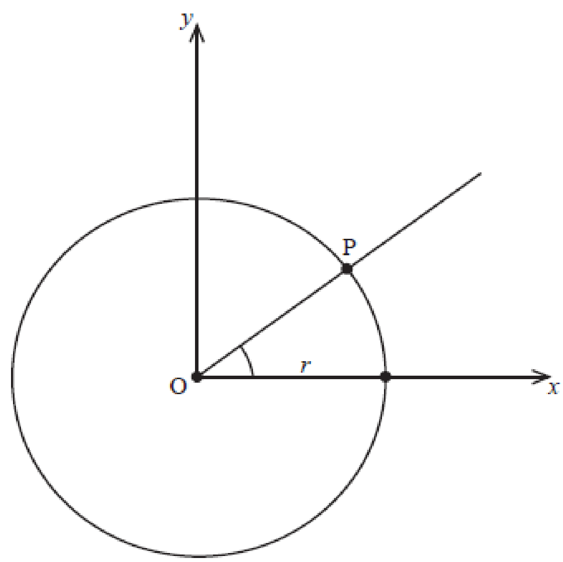
tangency point (1.66, –1.52)

[7 marks]

Examiners report

Very few candidates answered this question well but among those a variety of nice approaches were seen. This question required some organized thinking and good understanding of the concepts involved and therefore just strong candidates were able to go beyond the first steps. Sadly a few good answers were spoiled due to early rounding.

The diagram below shows a circle with centre at the origin O and radius $r > 0$.



A point P(x , y) , ($x > 0$, $y > 0$) is moving round the circumference of the circle.

Let $m = \tan\left(\arcsin \frac{y}{r}\right)$.

(a) Given that $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.001r$, show that $\frac{\mathrm{d}m}{\mathrm{d}t} = \left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$.

(b) State the geometrical meaning of $\frac{\mathrm{d}m}{\mathrm{d}t}$.

Markscheme

(a) $\frac{\mathrm{d}m}{\mathrm{d}t}=\frac{\mathrm{d}m}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$=\sec^2\left(\arcsin\frac{y}{r}\right)\times\left(\arcsin\frac{y}{r}\right)'\times\frac{r}{1000}$

$=\frac{1}{\cos^2\left(\arcsin\frac{y}{r}\right)}\times\frac{\frac{1}{r}}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\times\frac{r}{1000}$ (or equivalent) ***A1A1A1***

$=\frac{\frac{1}{\sqrt{r^2-y^2}}}{\frac{r^2-y^2}{r^2}}\frac{r}{1000}$ ***(A1)***

$=\frac{r^3}{10^3\sqrt{(r^2-y^2)^3}}$ (or equivalent) ***A1***

$=\left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$ ***AG N0***

(b) $\frac{\mathrm{d}m}{\mathrm{d}t}$ represents the rate of change of the gradient of the line OP ***A1***

[7 marks]

Examiners report

Few students were able to complete this question successfully, although many did obtain partial marks. Many students failed to recognise the difference between differentiating with respect to t or with respect to y . Very few were able to give a satisfactory geometrical meaning in part (b).

The cubic curve $y=8x^3+bx^2+cx+d$ has two distinct points P and Q, where the gradient is zero.

(a) Show that $b^2>24c$.

(b) Given that the coordinates of P and Q are $\left(\frac{1}{2},\ -12\right)$ and $\left(-\frac{3}{2},\ 20\right)$ respectively, find the values of b , c and d .

Markscheme

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}=24x^2+2bx+c$ ***(A1)***

$24x^2+2bx+c=0$ ***(M1)***

$\Delta=(2b)^2-96\,(c)$ ***(A1)***

$4b^2-96c>0$ ***A1***

$b^2>24c$ ***AG***

(b) $1+\frac{1}{4}b+\frac{1}{2}c+d=-12$

$6+b+c=0$

$-27+\frac{9}{4}b-\frac{3}{2}c+d=20$

$54-3b+c=0$ ***A1A1A1***

Note: Award ***A1*** for each correct equation, up to 3, not necessarily simplified.

$b = 12, c = -18, d = -7$ ***A1***

[8 marks]

Examiners report

Many candidates throughout almost the whole mark range were able to score well on this question. It was pleasing that most candidates were aware of the discriminant condition for distinct real roots of a quadratic. Some who dropped marks on part (b) either didn't write down a sufficient number of linear equations to determine the three unknowns or made arithmetic errors in their manual solution – few GDC solutions were seen.

Richard, a marine soldier, steps out of a stationary helicopter, 1000 m above the ground, at time $t = 0$. Let his height, in metres, above the ground be given by $s(t)$. For the first 10 seconds his velocity, $v(t)\mathrm{ms}^{-1}$, is given by $v(t) = -10t$.

a. (i) Find his acceleration $a(t)$ for $t < 10$. [6]

(ii) Calculate $v(10)$.

(iii) Show that $s(10) = 500$.

b. At $t = 10$ his parachute opens and his acceleration $a(t)$ is subsequently given by $a(t) = -10 - 5v,\ t \geq 10$. [1]

Given that $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{\frac{\mathrm{d}v}{\mathrm{d}t}}$, write down $\frac{\mathrm{d}t}{\mathrm{d}v}$ in terms of v.

c. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [5]

Hence show that $t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$.

d. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [2]

Hence find an expression for the velocity, v, for $t \geq 10$.

e. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [5]

Find an expression for his height, s, above the ground for $t \geq 10$.

f. You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive, for $t \geq 10$. [2]

Find the value of t when Richard lands on the ground.

Markscheme

a. (i) $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -10\ (\mathrm{ms}^{-2})$ ***A1***

(ii) $t = 10 \Rightarrow v = -100\ (\mathrm{ms}^{-1})$ ***A1***

(iii) $s = \int -10t\mathrm{d}t = -5t^2 (+c)$ ***M1A1***

$s = 1000 \text{ for } t = 0 \Rightarrow c = 1000$ ***(M1)***

$s = -5t^2 + 1000$ ***A1***

at $t = 10,\ s = 500\ (\mathrm{m})$ ***AG***

Note: Accept use of definite integrals.

[6 marks]

b. $\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{(-10-5v)}$ ***A1***

[1 mark]

c. **METHOD 1**

$t = \int \frac{1}{-10-5v}\mathrm{d}v = -\frac{1}{5}\ln(-10-5v)(+c)$ ***M1A1***

Note: Accept equivalent forms using modulus signs.

$t = 10,\ v = -100$

$10 = -\frac{1}{5}\ln(490) + c$ ***M1***

$c = 10 + \frac{1}{5}\ln(490)$ ***A1***

$t = 10 + \frac{1}{5}\ln 490 - \frac{1}{5}\ln(-10-5v)$ ***A1***

Note: Accept equivalent forms using modulus signs.

$t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

METHOD 2

$t = \int \frac{1}{-10-5v}\mathrm{d}v = -\frac{1}{5}\int \frac{1}{2+v}\mathrm{d}v = -\frac{1}{5}\ln|2+v|(+c)$ ***M1A1***

Note: Accept equivalent forms.

$t = 10,\ v = -100$

$10 = -\frac{1}{5}\ln|-98| + c$ ***M1***

Note: If $\ln(-98)$ is seen do not award further A marks.

$c = 10 + \frac{1}{5}\ln 98$ ***A1***

$t = 10 + \frac{1}{5}\ln 98 - \frac{1}{5}\ln|2+v|$ ***A1***

Note: Accept equivalent forms.

$t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

[5 marks]

d. $5(t-10) = \ln\frac{98}{(-2-v)}$

$\frac{2+v}{98} = -\mathrm{e}^{-5(t-10)}$ ***(M1)***

$v = -2 - 98\mathrm{e}^{-5(t-10)}$ ***A1***

[2 marks]

e. $\frac{\mathrm{d}s}{\mathrm{d}t} = -2 - 98\mathrm{e}^{-5(t-10)}$

$s = -2t + \frac{98}{5}\mathrm{e}^{-5(t-10)}(+k)$ ***M1A1***

at $t = 10,\ s = 500 \Rightarrow 500 = -20 + \frac{98}{5} + k \Rightarrow k = 500.4$ ***M1A1***

$s = -2t + \frac{98}{5}\mathrm{e}^{-5(t-10)} + 500.4$ ***A1***

Note: Accept use of definite integrals.

[5 marks]

f. $t = 250 \text{ for } s = 0$ ***(M1)A1***

[2 marks]

Total [21 marks]

Examiners report

a. Parts (i) and (ii) were well answered by most candidates.

In (iii) the constant of integration was often forgotten. Most candidates calculated the displacement and then used different strategies, mostly incorrect, to remove the negative sign from -500.

b. Surprisingly part (b) was not well done as the question stated the method. Many candidates simply wrote down $\frac{\mathrm{d}v}{\mathrm{d}t}$ while others seemed unaware that $\frac{\mathrm{d}v}{\mathrm{d}t}$ was the acceleration.

c. Part (c) was not always well done as it followed from (b) and at times there was very little to allow follow through. Once again some candidates started with what they were trying to prove. Among the candidates that attempted to integrate many did not consider the constant of integration properly.

d. In part (d) many candidates ignored the answer given in (c) and attempted to manipulate different expressions.

e. Part (e) was poorly answered: the constant of integration was often again forgotten and some inappropriate uses of Physics formulas assuming that the acceleration was constant were used. There was unclear thinking with the two sides of an equation being integrated with respect to different variables.

f. Although part (e) was often incorrect, some follow through marks were gained in part (f).

The curve C is defined by equation $xy - \ln y = 1,\ y > 0$.

a. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

b. Determine the equation of the tangent to C at the point $\left(\frac{2}{\mathrm{e}},\ \mathrm{e}\right)$ [3]

Markscheme

a. $y + x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1A1A1***

Note: Award ***A1*** for the first two terms, ***A1*** for the third term and the 0.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{1-xy}$ ***A1***

Note: Accept $\frac{-y^2}{\ln y}$.

Note: Accept $\frac{-y}{x-\frac{1}{y}}$.

[4 marks]

b. $m_T = \frac{\mathrm{e}^2}{1-\mathrm{e}\times\frac{2}{\mathrm{e}}}$ ***(M1)***

$m_T = -\mathrm{e}^2$ ***(A1)***

$y - \mathrm{e} = -\mathrm{e}^2 x + 2\mathrm{e}$

$-\mathrm{e}^2 x - y + 3\mathrm{e} = 0$ or equivalent ***A1***

Note: Accept $y = -7.39x + 8.15$.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

A skydiver jumps from a stationary balloon at a height of 2000 m above the ground.

Her velocity, $v\ \mathrm{ms}^{-1}$, t seconds after jumping, is given by $v = 50(1 - \mathrm{e}^{-0.2t})$.

a. Find her acceleration 10 seconds after jumping. [3]

b. How far above the ground is she 10 seconds after jumping? [3]

Markscheme

a. $a = 10\mathrm{e}^{-0.2t}$ ***(M1)(A1)***

at $t = 10$, $a = 1.35\ (\mathrm{ms}^{-2})\quad (\text{accept } 10e^{-2})$ ***A1***

[3 marks]

b. **METHOD 1**

$d = \int_0^{10} 50(1 - \mathrm{e}^{-0.2t})\mathrm{d}t$ ***(M1)***

$= 283.83\ldots$ ***A1***

so distance above ground $= 1720\ (\mathrm{m})\ (3\ \mathrm{sf})\ \ (\text{accept } 1716\ (\mathrm{m}))$ ***A1***

METHOD 2

$\mathrm{s} = \int 50(1 - \mathrm{e}^{-0.2t})\mathrm{d}t = 50t + 250\mathrm{e}^{-0.2t}(+c)$ ***M1***

Taking $s = 0$ when $t = 0$ gives $c = -250$ ***M1***

So when $t = 10$, $s = 283.3\ldots$

so distance above ground $= 1720\ (\mathrm{m})\ (3\ \mathrm{sf})\ \ (\text{accept } 1716\ (\mathrm{m}))$ ***A1***

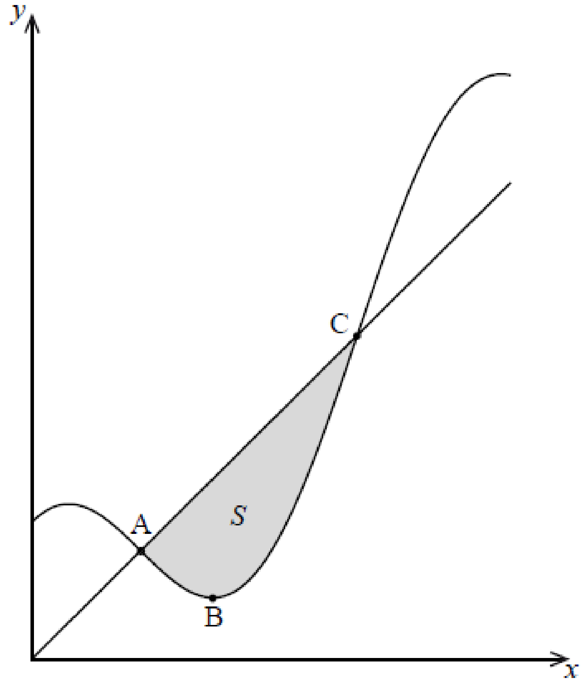
[3 marks]

Examiners report

a. Part (a) was generally correctly answered. A few candidates suffered the Arithmetic Penalty for giving their answer to more than 3sf. A smaller number were unable to differentiate the exponential function correctly. Part (b) was less well answered, many candidates not thinking clearly about the position and direction associated with the initial conditions.

b. Part (a) was generally correctly answered. A few candidates suffered the Arithmetic Penalty for giving their answer to more than 3sf. A smaller number were unable to differentiate the exponential function correctly. Part (b) was less well answered, many candidates not thinking clearly about the position and direction associated with the initial conditions.

Let f be a function defined by $f(x) = x + 2\cos x$, $x \in [0,\ 2\pi]$. The diagram below shows a region S bound by the graph of f and the line $y = x$.



A and C are the points of intersection of the line $y = x$ and the graph of f , and B is the minimum point of f .

(a) If A, B and C have x-coordinates $a\frac{\pi}{2}$, $b\frac{\pi}{6}$ and $c\frac{\pi}{2}$, where a , b, $c \in \mathbb{N}$, find the values of a , b and c .

(b) Find the range of f .

(c) Find the equation of the normal to the graph of f at the point C, giving your answer in the form $y = px + q$.

(d) The region S is rotated through 2π about the x-axis to generate a solid.

(i) Write down an integral that represents the volume V of this solid.

(ii) Show that $V = 6\pi^2$.

Markscheme

(a) **METHOD 1**

using GDC

$a = 1, b = 5, c = 3$ ***A1A2A1***

METHOD 2

$x = x + 2\cos x \Rightarrow \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$ ***M1***

$a = 1, c = 3$ ***A1***

$1 - 2\sin x = 0$ ***M1***

$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$b = 5$ ***A1***

Note: Final ***M1A1*** is independent of previous work.

[4 marks]

(b) $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3}$ (or 0.886) ***(M1)***

$f(2\pi) = 2\pi + 2$ (or 8.28) ***(M1)***

the range is $\left[\frac{5\pi}{6} - \sqrt{3},\ 2\pi + 2\right]$ (or [0.886, 8.28]) ***A1***

[3 marks]

(c) $f'(x) = 1 - 2\sin x$ ***(M1)***

$f'\left(\frac{3\pi}{6}\right) = 3$ ***A1***

gradient of normal $= -\frac{1}{3}$ ***(M1)***

equation of the normal is $y - \frac{3\pi}{2} = -\frac{1}{3}\left(x - \frac{3\pi}{2}\right)$ ***(M1)***

$y = -\frac{1}{3}x + 2\pi$ (or equivalent decimal values) ***A1*** ***N4***

[5 marks]

(d) (i) $V = \pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(x^2 - (x + 2\cos x)^2\right) \mathrm{d}x$ (or equivalent) ***A1A1***

Note: Award ***A1*** for limits and ***A1*** for π and integrand.

(ii) $V = \pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(x^2 - (x + 2\cos x)^2\right) \mathrm{d}x$

$= -\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(4x\cos x + 4\cos^2 x\right) \mathrm{d}x$

using integration by parts ***M1***

and the identity $4\cos^2 x = 2\cos 2x + 2$, ***M1***

$V = -\pi[(4x\sin x + 4\cos x) + (\sin 2x + 2x)]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ ***A1A1***

Note: Award ***A1*** for $4x\sin x + 4\cos x$ and ***A1*** for $\sin 2x + 2x$.

$= -\pi\left[\left(6\pi\sin\frac{3\pi}{2} + 4\cos\frac{3\pi}{2} + \sin 3\pi + 3\pi\right) - \left(2\pi sin\frac{\pi}{2} + 4\cos\frac{\pi}{2} + \sin\pi + \pi\right)\right]$ ***A1***

$= -\pi\left(-6\pi + 3\pi - \pi\right)$

$= 6\pi^2$ ***AG*** ***N0***

Note: Do not accept numerical answers.

[7 marks]

Total [19 marks]

Examiners report

Generally there were many good attempts to this, more difficult, question. A number of students found b to be equal to 1, rather than 5. In the final part few students could successfully work through the entire integral successfully.

A curve C is given by the implicit equation $x + y - \cos(xy) = 0$.

The curve $xy = -\frac{\pi}{2}$ intersects C at P and Q.

a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{1+y\sin(xy)}{1+x\sin(xy)}\right)$. [5]

b.i. Find the coordinates of P and Q. [4]

b.ii. Given that the gradients of the tangents to C at P and Q are m_1 and m_2 respectively, show that $m_1 \times m_2 = 1$. [3]

c. Find the coordinates of the three points on C, nearest the origin, where the tangent is parallel to the line $y = -x$. [7]

Markscheme

a. attempt at implicit differentiation ***M1***

$1 + \frac{\mathrm{d}y}{\mathrm{d}x} + \left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\sin(xy) = 0$ ***A1M1A1***

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{\mathrm{d}y}{\mathrm{d}x} = -1 - y\sin(xy)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{1+y\sin(xy)}{1+x\sin(xy)}\right)$ ***AG***

[5 marks]

b.i. **EITHER**

when $xy = -\frac{\pi}{2},\ \cos xy = 0$ ***M1***

$\Rightarrow x + y = 0$ ***(A1)***

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent ***M1***

$x - \frac{\pi}{2x} = 0$ ***(A1)***

THEN

therefore $x^2 = \frac{\pi}{2}\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ ***A1***

$\mathrm{P}\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right),\ \mathrm{Q}\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ **or** $P(1.25, -1.25), Q(-1.25, 1.25)$ ***A1***

[4 marks]

b.ii. $m_1 = -\left(\frac{1-\sqrt{\frac{\pi}{2}}\times -1}{1+\sqrt{\frac{\pi}{2}}\times -1}\right)$ ***M1A1***

$m_2 = -\left(\frac{1+\sqrt{\frac{\pi}{2}}\times -1}{1-\sqrt{\frac{\pi}{2}}\times -1}\right)$ ***A1***

$m_1 m_2 = 1$ ***AG***

Note: Award ***M1A0A0*** if decimal approximations are used.
Note: No **FT** applies.

[3 marks]

c. equate derivative to −1 ***M1***

$(y-x)\sin(xy)=0$ ***(A1)***

$y=x,\ \sin(xy)=0$ ***R1***

in the first case, attempt to solve $2x=\cos\left(x^2\right)$ ***M1***

(0.486,0.486) **A1**

in the second case, $\sin(xy)=0\Rightarrow xy=0$ and $x+y=1$ ***(M1)***

(0,1), (1,0) ***A1***

[7 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c. [N/A]

Consider the curve with equation $\left(x^2+y^2\right)^2=4xy^2$.

a. Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [5]

b. Find the equation of the normal to the curve at the point (1, 1). [3]

Markscheme

a. **METHOD 1**

expanding the brackets first:
$x^4+2x^2y^2+y^4=4xy^2$ ***M1***
$4x^3+4xy^2+4x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+4y^3\frac{\mathrm{d}y}{\mathrm{d}x}=4y^2+8xy\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1***

Note: Award ***M1*** for an attempt at implicit differentiation.
Award ***A1*** for each side correct.

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x^3-xy^2+y^2}{xy^2-2xy+y^3}$ or equivalent ***A1***
METHOD 2
$2\left(x^2+y^2\right)\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)=4y^2+8xy\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1***

Note: Award ***M1*** for an attempt at implicit differentiation.
Award ***A1*** for each side correct.

$\left(x^2+y^2\right)\left(x+y\frac{\mathrm{d}y}{\mathrm{d}x}\right)=y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}$
$x^3+x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2x+y^3\frac{\mathrm{d}y}{\mathrm{d}x}=y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1***
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-x^3-xy^2+y^2}{yx^2-2xy+y^3}$ or equivalent ***A1***
[5 marks]

b. **METHOD 1**

at (1, 1), $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined ***M1A1***

$y = 1$ ***A1***

METHOD 2

gradient of normal $= -\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = -\frac{(yx^2-2xy+y^3)}{(-x^3-xy^2+y^2)}$ ***M1***

at (1, 1) gradient $= 0$ ***A1***

$y = 1$ ***A1***

[3 marks]

Examiners report

a. [N/A]

b. [N/A]

An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, 2π radians about the y-axis. Units on the coordinate axes are defined to be in centimetres.

a. When the glass contains water to a height h cm, find the volume V of water in terms of h . [3]

b. If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that, [6]

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$, where t is measured in hours.

c. If the glass is filled completely, how long will it take for all the water to evaporate? [7]

Markscheme

a. volume $= \pi \int_0^h x^2 \mathrm{d}y$ ***(M1)***

$\pi \int_0^h y\mathrm{d}y$ ***M1***

$\pi \left[\frac{y^2}{2}\right]_0^h = \frac{\pi h^2}{2}$ ***A1***

[3 marks]

b. $\frac{\mathrm{d}V}{\mathrm{d}t} = -3\times$ surface area ***A1***

surface area $= \pi x^2$ ***(M1)***

$= \pi h$ ***A1***

$V = \frac{\pi h^2}{2} \Rightarrow h\sqrt{\frac{2V}{\pi}}$ ***M1A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\pi\sqrt{\frac{2V}{\pi}}$ ***A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$ ***AG***

Note: Assuming that $\frac{\mathrm{d}h}{\mathrm{d}t} = -3$ without justification gains no marks.

[6 marks]

c. $V_0 = 5000\pi$ ($= 15700$ cm^3) ***A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$

attempting to separate variables ***M1***

EITHER

$\int \frac{\mathrm{d}V}{\sqrt{V}} = -3\sqrt{2\pi}\int \mathrm{d}t$ ***A1***

$2\sqrt{V} = -3\sqrt{2\pi}t + c$ ***A1***

$c = 2\sqrt{5000\pi}$ ***A1***

$V = 0$ ***M1***

$\Rightarrow t = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours ***A1***

OR

$\int_{5000\pi}^{0} \frac{\mathrm{d}V}{\sqrt{V}} = -3\sqrt{2\pi}\int_0^T \mathrm{d}t$ ***M1A1A1***

Note: Award ***M1*** for attempt to use definite integrals, ***A1*** for correct limits and ***A1*** for correct integrands.

$\left[2\sqrt{V}\right]_{5000\pi}^{0} = 3\sqrt{2\pi}T$ ***A1***

$T = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours ***A1***

[7 marks]

Examiners report

a. This question was found to be challenging by many candidates and there were very few completely correct solutions. Many candidates did not seem able to find the volume of revolution when taken about the *y*-axis in (a). Candidates did not always recognize that part (b) did not involve related rates. Those candidates who attempted the question made some progress by separating the variables and integrating in (c) but very few were able to identify successfully the values necessary to find the correct answer.

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Xavier, the parachutist, jumps out of a plane at a height of h metres above the ground. After free falling for 10 seconds his parachute opens. His velocity, $v\,\mathrm{ms}^{-1}$, t seconds after jumping from the plane, can be modelled by the function

$$v(t)=\begin{cases} 9.8t, & 0\leqslant t\leqslant 10 \\ \frac{98}{\sqrt{1+(t-10)^2}}, & t>10 \end{cases}$$

His velocity when he reaches the ground is $2.8\ \mathrm{ms}^{-1}$.

a. Find his velocity when $t=15$. [2]

b. Calculate the vertical distance Xavier travelled in the first 10 seconds. [2]

c. Determine the value of h. [5]

Markscheme

a. $v(15)=\frac{98}{\sqrt{1+(15-10)^2}}$ ***(M1)***

$v(15)=19.2\ (\mathrm{ms}^{-1})$ ***A1***

[2 marks]

b. $\int_0^{10} 9.8t\,\mathrm{d}t$ ***(M1)***

$=490\ (\mathrm{m})$ ***A1***

[2 marks]

c. $\frac{98}{\sqrt{1+(t-10)^2}}=2.8$ ***(M1)***

$t=44.985\ldots\ (\mathrm{s})$ ***A1***

$h=490+\int_{10}^{44.9\ldots}\frac{98}{\sqrt{1+(t-10)^2}}\mathrm{d}t$ ***(M1)(A1)***

$h=906\ (\mathrm{m})$ ***A1***

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The function $f(x)=3\sin x+4\cos x$ is defined for $0<x<2\pi$.

a. Write down the coordinates of the minimum point on the graph of f. [1]

b. The points $\mathrm{P}(p,\ 3)$ and $\mathrm{Q}(q,\ 3)$, $q>p$, lie on the graph of $y=f(x)$. [2]

Find p and q .

c. Find the coordinates of the point, on $y = f(x)$, where the gradient of the graph is 3. [4]

d. Find the coordinates of the point of intersection of the normals to the graph at the points P and Q. [7]

Markscheme

a. $(3.79, -5)$ ***A1***

[1 mark]

b. $p = 1.57$ or $\frac{\pi}{2}$, $q = 6.00$ ***A1A1***

[2 marks]

c. $f'(x) = 3\cos x - 4\sin x$ ***(M1)(A1)***

$3\cos x - 4\sin x = 3 \Rightarrow x = 4.43...$ ***(A1)***

$(y = -4)$ ***A1***

Coordinates are $(4.43, -4)$

[4 marks]

d. $m_{\text{normal}} = \frac{1}{m_{\text{tangent}}}$ ***(M1)***

gradient at P is -4 so gradient of normal at P is $\frac{1}{4}$ ***(A1)***

gradient at Q is 4 so gradient of normal at Q is $-\frac{1}{4}$ ***(A1)***

equation of normal at P is $y - 3 = \frac{1}{4}(x - 1.570...)$ (or $y = 0.25x + 2.60...$) ***(M1)***

equation of normal at Q is $y - 3 = \frac{1}{4}(x - 5.999...)$ (or $y = -0.25x + \underbrace{4.499...}$) ***(M1)***

Note: Award the previous two ***M1*** even if the gradients are incorrect in $y - b = m(x - a)$ where (a, b) are coordinates of P and Q (or in $y = mx + c$ with c determined using coordinates of P and Q.

intersect at $(3.79,\ 3.55)$ ***A1A1***

Note: Award ***N2*** for 3.79 without other working.

[7 marks]

Examiners report

a. Candidates answered parts (a) and (b) of this question well and, although many were also successful in part (c), just a few candidates gave answers to the required level of accuracy. Part d) was rather challenging for many candidates. The most common errors among the candidates who attempted this question were the confusion between tangents and normals and incorrect final answers due to premature rounding.

b. Candidates answered parts (a) and (b) of this question well and, although many were also successful in part (c), just a few candidates gave answers to the required level of accuracy. Part d) was rather challenging for many candidates. The most common errors among the candidates who attempted this question were the confusion between tangents and normals and incorrect final answers due to premature rounding.

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Find the equation of the normal to the curve $x^3y^3 - xy = 0$ at the point (1, 1).

Markscheme

$x^3y^3 - xy = 0$

$3x^2y^3 + 3x^3y^2y' - y - xy' = 0$

Note: Award ***A1*** for correctly differentiating each term.

$x = 1,\ y = 1 \quad 3 + 3y' - 1 - y' = 0$

$2y' = -2$

$y' = -1$ ***(M1)A1***

gradient of normal = 1 ***(A1)***

equation of the normal $y - 1 = x - 1$ ***A1*** ***N2***

$y = x$

Note: Award ***A2R5*** for correct answer and correct justification.

[7 marks]

Examiners report

This implicit differentiation question was well answered by most candidates with many achieving full marks. Some candidates made algebraic errors which prevented them from scoring well in this question.

Other candidates realised that the equation of the curve could be simplified although the simplification was seldom justified.

A curve is defined $x^2 - 5xy + y^2 = 7$.

a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y-2x}{2y-5x}$. [3]

b. Find the equation of the normal to the curve at the point $(6,\ 1)$. [4]

c. Find the distance between the two points on the curve where each tangent is parallel to the line $y = x$. [8]

Markscheme

a. attempt at implicit differentiation ***M1***

$2x - 5x\frac{\mathrm{d}y}{\mathrm{d}x} - 5y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1A1***

Note: ***A1*** for differentiation of $x^2 - 5xy$, ***A1*** for differentiation of y^2 and 7.

$2x - 5y + \frac{\mathrm{d}y}{\mathrm{d}x}(2y - 5x) = 0$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y-2x}{2y-5x}$ ***AG***

[3 marks]

b. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5\times1-2\times6}{2\times1-5\times6} = \frac{1}{4}$ ***A1***

gradient of normal $= -4$ ***A1***

equation of normal $y = -4x + c$ ***M1***

substitution of $(6,\ 1)$

$y = -4x + 25$ ***A1***

Note: Accept $y - 1 = -4(x - 6)$

[4 marks]

c. setting $\frac{5y-2x}{2y-5x} = 1$ ***M1***

$y = -x$ ***A1***

substituting into original equation ***M1***

$x^2 + 5x^2 + x^2 = 7$ ***(A1)***

$7x^2 = 7$

$x = \pm 1$ ***A1***

points $(1,\ -1)$ and $(-1,\ 1)$ ***(A1)***

distance $= \sqrt{8}$ $\left(= 2\sqrt{2}\right)$ ***(M1)A1***

[8 marks]

Total [15 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The point P, with coordinates $(p,\ q)$, lies on the graph of $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, $a>0$.

The tangent to the curve at P cuts the axes at (0, m) and (n, 0) . Show that $m+n=a$.

Markscheme

$x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$

$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}=-\sqrt{\frac{y}{x}}$ ***A1***

Note: Accept $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$ from making y the subject of the equation, and all correct subsequent working

therefore the gradient at the point P is given by

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{\frac{q}{p}}$ ***A1***

equation of tangent is $y-q=-\sqrt{\frac{q}{p}}(x-p)$ ***M1***

$\left(y=-\sqrt{\frac{q}{p}}x+q+\sqrt{q}\sqrt{p}\right)$

x-intercept: $y=0$, $n=\frac{q\sqrt{p}}{\sqrt{q}}+p=\sqrt{q}\sqrt{p}+p$ ***A1***

y-intercept: $x=0$, $m=\sqrt{q}\sqrt{p}+q$ ***A1***

$n+m=\sqrt{q}\sqrt{p}+p+\sqrt{q}\sqrt{p}+q$ ***M1***

$=2\sqrt{q}\sqrt{p}+p+q$

$=\left(\sqrt{p}+\sqrt{q}\right)^{2}$ ***A1***

$=a$ ***AG***

[8 marks]

Examiners report

Many candidates were able to perform the implicit differentiation. Few gained any further marks.

Consider the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos 2x$.

a. (i) Show that the function $y=\cos x+\sin x$ satisfies the differential equation. [10]

(ii) Find the general solution of the differential equation. Express your solution in the form $y=f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)?

b. A different solution of the differential equation, satisfying $y=2$ when $x=\frac{\pi}{4}$, defines a curve C. [12]

(i) Determine the equation of C in the form $y=g(x)$, and state the range of the function g.

A region R in the xy plane is bounded by C, the x-axis and the vertical lines $x=0$ and $x=\frac{\pi}{2}$.

(ii) Find the area of R.

(iii) Find the volume generated when that part of R above the line $y = 1$ is rotated about the x-axis through 2π radians.

Markscheme

a. (i) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x + \cos x$ ***A1***

$y\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x + \sin x)(-\sin x + \cos x)$ ***M1***

$= \cos^2 x - \sin^2 x$ ***A1***

$= \cos 2x$ ***AG***

METHOD 2

$y^2 = (\sin x + \cos x)^2$ ***A1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\cos x + \sin x)(\cos x - \sin x)$ ***M1***

$y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 x - \sin^2 x$ ***A1***

$= \cos 2x$ ***AG***

(ii) attempting to separate variables $\int y\,\mathrm{d}y = \int \cos 2x\,\mathrm{d}x$ ***M1***

$\frac{1}{2}y^2 = \frac{1}{2}\sin 2x + C$ ***A1A1***

Note: Award ***A1*** for a correct LHS and ***A1*** for a correct RHS.

$y = \pm(\sin 2x + A)^{\frac{1}{2}}$ ***A1***

(iii) $\sin 2x + A \equiv (\cos x + \sin x)^2$ ***(M1)***

$(\cos x + \sin x)^2 = \cos^2 x + 2\sin x\cos x + \sin^2 x$

use of $\sin 2x \equiv 2\sin x\cos x$. ***(M1)***

$A = 1$ ***A1***

[10 marks]

b. (i) substituting $x = \frac{\pi}{4}$ and $y = 2$ into $y = (\sin 2x + A)^{\frac{1}{2}}$ ***M1***

so $g(x) = (\sin 2x + 3)^{\frac{1}{2}}$. ***A1***

range g is $\left[\sqrt{2},\ 2\right]$ ***A1A1A1***

Note: Accept [1.41, 2]. Award ***A1*** for each correct endpoint and ***A1*** for the correct closed interval.

(ii) $\int_0^{\frac{\pi}{2}} (\sin 2x + 3)^{\frac{1}{2}}\mathrm{d}x$ ***(M1)(A1)***

$= 2.99$ ***A1***

(iii) $\pi \int_0^{\frac{\pi}{2}} (\sin 2x + 3)\mathrm{d}x - \pi(1)\left(\frac{\pi}{2}\right)$ (or equivalent) ***(M1)(A1)(A1)***

Note: Award ***(M1)(A1)(A1)*** for $\pi \int_0^{\frac{\pi}{2}} (\sin 2x + 2)\mathrm{d}x$

$= 17.946 - 4.935$ $\left(= \frac{\pi}{2}(3\pi + 2) - \pi\left(\frac{\pi}{2}\right)\right)$ ***A1***

Note: Award ***A1*** for $\pi(\pi + 1)$.

[12 marks]

Examiners report

a. Part (a) was not well done and was often difficult to mark. In part (a) (i), a large number of candidates did not know how to verify a solution, $y(x)$, to the given differential equation. Instead, many candidates attempted to solve the differential equation. In part (a) (ii), a large number of candidates began solving the differential equation by correctly separating the variables but then either neglected to add a constant of integration or added one as an afterthought. Many simple algebraic and basic integral calculus errors were seen. In part (a) (iii), many candidates did not realize that the solution given in part (a) (i) and the general solution found in part (a) (ii) were to be equated. Those that did know to equate these two solutions, were able to square both solution forms and correctly use the trigonometric identity $\sin 2x = 2\sin x \cos x$. Many of these candidates however started with incorrect solution(s).

b. In part (b), a large number of candidates knew how to find a required area and a required volume of solid of revolution using integral calculus. Many candidates, however, used incorrect expressions obtained in part (a). In part (b) (ii), a number of candidates either neglected to state 'π' or attempted to calculate the volume of a solid of revolution of 'radius' $f(x) - g(x)$.

By using the substitution $x = 2\tan u$, show that $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}} = \frac{-\sqrt{x^2+4}}{4x} + C$.

Markscheme

EITHER

$\frac{\mathrm{d}x}{\mathrm{d}u} = 2\sec^2 u$ ***A1***

$\int \frac{2\sec^2 u\mathrm{d}u}{4\tan^2 u\sqrt{4+4\tan^2 u}}$ ***(M1)***

$\int \frac{2\sec^2 u\mathrm{d}u}{4\tan^2 u \times 2\sec u}$ $\left(= \int \frac{\mathrm{d}u}{4\sin^2 u\sqrt{\tan^2 u+1}} \text{ or } = \int \frac{2\sec^2 u\mathrm{d}u}{4\tan^2 u\sqrt{4\sec^2 u}}\right)$ ***A1***

OR

$u = \arctan\frac{x}{2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x^2+4}$ ***A1***

$\int \frac{\sqrt{4\tan^2 u+4}\mathrm{d}u}{2\times 4\tan^2 u}$ ***(M1)***

$\int \frac{2\sec u\mathrm{d}u}{2\times 4\tan^2 u}$ ***A1***

THEN

$= \frac{1}{4}\int \frac{\sec u\mathrm{d}u}{\tan^2 u}$

$= \frac{1}{4}\int \operatorname{cosec} u \cot u\mathrm{d}u$ $\left(= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\mathrm{d}u\right)$ ***A1***

$= -\frac{1}{4}\operatorname{cosec} u(+C)\ \left(= -\frac{1}{4\sin u}(+C)\right)$ ***A1***

use of either $u = \frac{x}{2}$ or an appropriate trigonometric identity ***M1***

either $\sin u = \frac{x}{\sqrt{x^2+4}}$ or $\operatorname{cosec} u = \frac{\sqrt{x^2+4}}{x}$ (or equivalent) ***A1***

$= \frac{-\sqrt{x^2+4}}{4x}(+C)$ ***AG***

[7 marks]

Examiners report

Most candidates found this a challenging question. A large majority of candidates were able to change variable from *x* to *u* but were not able to make any further progress.

The function *f* is defined by $f(x) = x\sqrt{9-x^2} + 2\arcsin\left(\frac{x}{3}\right)$.

(a) Write down the largest possible domain, for each of the two terms of the function, f, and hence state the largest possible domain, D, for f.

(b) Find the volume generated when the region bounded by the curve $y = f(x)$, the *x*-axis, the *y*-axis and the line $x = 2.8$ is rotated through 2π radians about the *x*-axis.

(c) Find $f'(x)$ in simplified form.

(d) **Hence** show that $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where $p \in D$.

(e) Find the value of p which maximises the value of the integral in (d).

(f) (i) Show that $f''(x) = \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$.

(ii) Hence justify that $f(x)$ has a point of inflexion at $x = 0$, but not at $x = \pm\sqrt{\frac{25}{2}}$.

Markscheme

(a) For $x\sqrt{9-x^2}$, $-3 \leqslant x \leqslant 3$ and for $2\arcsin\left(\frac{x}{3}\right)$, $-3 \leqslant x \leqslant 3$ ***A1***

$\Rightarrow D$ is $-3 \leqslant x \leqslant 3$ ***A1***

[2 marks]

(b) $V = \pi\int_0^{2.8}\left(x\sqrt{9-x^2} = 2\arcsin\frac{x}{3}\right)^2\mathrm{d}x$ ***M1A1***

= 181 ***A1***

[3 marks]

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} = (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{\frac{2}{3}}{\sqrt{1-\frac{x^2}{9}}}$ ***M1A1***

$= (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{2}{(9-x^2)^{\frac{1}{2}}}$ ***A1***

$= \frac{9-x^2-x^2+2}{(9-x^2)^{\frac{1}{2}}}$ ***A1***

$= \frac{11-2x^2}{\sqrt{9-x^2}}$ ***A1***

[5 marks]

(d) $\int_{-p}^{p}\frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = \left[x\sqrt{9-x^2} + 2\arcsin\frac{x}{3}\right]_{-p}^{p}$ ***M1***

$= p\sqrt{9-p^2} + 2\arcsin\frac{p}{3} + p\sqrt{9-p^2} + 2\arcsin\frac{p}{3}$ ***A1***

$= 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$ ***AG***

[2 marks]

(e) $11 - 2p^2 = 0$ ***M1***

$p = 2.35$ $\left(\sqrt{\frac{11}{2}}\right)$ ***A1***

Note: Award ***A0*** for $p = \pm 2.35$.

[2 marks]

(f) (i) $f''(x) = \frac{(9-x^2)^{\frac{1}{2}}(-4x)+x(11-2x^2)(9-x^2)^{-\frac{1}{2}}}{9-x^2}$ ***M1A1***

$= \frac{-4x(9-x^2)+x(11-2x^2)}{(9-x^2)^{\frac{3}{2}}}$ ***A1***

$= \frac{-36x+4x^3+11x-2x^3}{(9-x^2)^{\frac{3}{2}}}$ ***A1***

$= \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$ ***AG***

(ii) **EITHER**

When $0 < x < 3$, $f''(x) < 0$. When $-3 < x < 0$, $f''(x) > 0$. ***A1***

OR

$f''(0) = 0$ ***A1***

THEN

Hence $f''(x)$ changes sign through $x = 0$, giving a point of inflexion. ***R1***

EITHER

$x = \pm\sqrt{\frac{25}{2}}$ is outside the domain of *f*. ***R1***

OR

$x = \pm\sqrt{\frac{25}{2}}$ is not a root of $f''(x) = 0$. ***R1***

[7 marks]

Total [21 marks]

Examiners report

It was disappointing to note that some candidates did not know the domain for arcsin. Most candidates knew what to do in (b) but sometimes the wrong answer was obtained due to the calculator being in the wrong mode. In (c), the differentiation was often disappointing with $\arcsin\left(\frac{x}{3}\right)$ causing problems. In (f)(i), some candidates who failed to do (c) guessed the correct form of $f'(x)$ (presumably from (d)) and then went on to find $f''(x)$ correctly. In (f)(ii), the justification of a point of inflexion at $x = 0$ was sometimes incorrect – for example, some candidates showed simply that $f'(x)$ is positive on either side of the origin which is not a valid reason.

A body is moving through a liquid so that its acceleration can be expressed as

$$\left(-\frac{v^2}{200}-32\right)\ \mathrm{ms}^{-2},$$

where v ms^{-1} is the velocity of the body at time t seconds.

The initial velocity of the body was known to be 40 ms^{-1}.

(a) Show that the time taken, T seconds, for the body to slow to V ms^{-1} is given by

$$T=200\int_V^{40}\frac{1}{v^2+80^2}\mathrm{d}v.$$

(b) (i) Explain why acceleration can be expressed as $v\frac{\mathrm{d}v}{\mathrm{d}s}$, where s is displacement, in metres, of the body at time t seconds.

(ii) **Hence** find a similar integral to that shown in part (a) for the distance, S metres, travelled as the body slows to V ms^{-1}.

(c) **Hence**, using parts (a) and (b), find the distance travelled and the time taken until the body momentarily comes to rest.

Markscheme

(a) $\frac{\mathrm{d}v}{\mathrm{d}t}=-\frac{v^2}{200}-32\left(=\frac{-v^2-6400}{200}\right)$ ***(M1)***

$\int_0^T \mathrm{d}t=\int_{40}^V-\frac{200}{v^2+80^2}\mathrm{d}v$ ***M1A1A1***

$T=200\int_V^{40}\frac{1}{v^2+80^2}\mathrm{d}v$ ***AG***

[4 marks]

(b) (i) $a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}s}\times\frac{\mathrm{d}s}{\mathrm{d}t}$ ***R1***

$=v\frac{\mathrm{d}v}{\mathrm{d}s}$ ***AG***

(ii) $v\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{-v^2-80^2}{200}$ ***(M1)***

$\int_0^S \mathrm{d}s=\int_{40}^V-\frac{200v}{v^2+80^2}\mathrm{d}v$ ***M1A1A1***

$\int_0^S \mathrm{d}s=\int_V^{40}\frac{200v}{v^2+80^2}\mathrm{d}v$ ***M1***

$S=200\int_V^{40}\frac{v}{v^2+80^2}\mathrm{d}v$ ***A1***

[7 marks]

(c) letting $V=0$ ***(M1)***

distance $=200\int_0^{40}\frac{v}{v^2+80^2}\mathrm{d}v=22.3$ metres ***A1***

time $=200\int_0^{40}\frac{1}{v^2+80^2}\mathrm{d}v=1.16$ seconds ***A1***

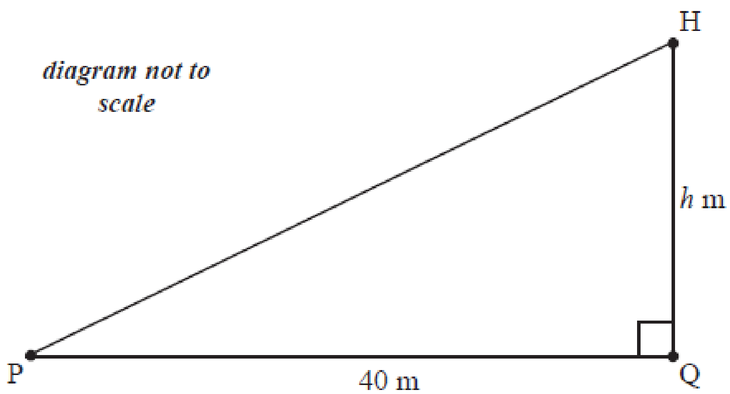
[3 marks]

Total [14 marks]

Examiners report

Many students failed to understand the problem as one of solving differential equations. In addition there were many problems seen in finding the end points for the definite integrals. Part (b) (i) should have been a simple point having used the chain rule, but it seemed that many students had not seen this, even though it is clearly in the syllabus.

A helicopter H is moving vertically upwards with a speed of 10 ms^{-1}. The helicopter is h m directly above the point Q which is situated on level ground. The helicopter is observed from the point P which is also at ground level and PQ $= 40$ m. This information is represented in the diagram below.



When $h = 30$,

(a) show that the rate of change of $\text{H}\hat{\text{P}}\text{Q}$ is 0.16 radians per second;

(b) find the rate of change of PH.

Markscheme

(a) let $\text{H}\hat{\text{P}}\text{Q} = \theta$

$\tan\theta = \frac{h}{40}$

$\sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{40}\frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{4\sec^2\theta}$ ***(A1)***

$= \frac{16}{4\times 25}$ $\left(\sec\theta = \frac{5}{4} \text{ or } \theta = 0.6435\right)$ ***A1***

$= 0.16$ radians per second ***AG***

(b) $x^2 = h^2 + 1600$, where PH $= x$

$2x\frac{\mathrm{d}x}{\mathrm{d}t} = 2h\frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{h}{x} \times 10$ ***A1***

$= \frac{10h}{\sqrt{h^2+1600}}$ ***(A1)***

$h = 30$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 6 \text{ ms}^{-1}$ ***A1***

Note: Accept solutions that begin $x = 40\sec\theta$ or use $h = 10t$.

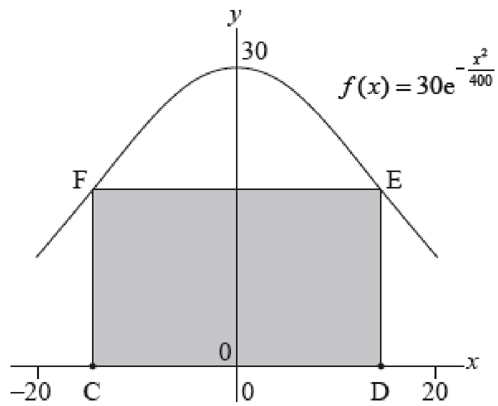
[7 marks]

Examiners report

For those candidates who realized this was an applied calculus problem involving related rates of change, the main source of error was in differentiating inverse tan in part (a). Some found part (b) easier than part (a), involving a changing length rather than an angle. A number of alternative approaches were reported by examiners.

The following diagram shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $f(x) = 30\mathrm{e}^{-\frac{x^2}{400}}$, where $-20 \le x \le 20$.

Ground level is represented by the x-axis.



a. Find $f''(x)$. [4]

b. Show that the gradient of the roof function is greatest when $x = -\sqrt{200}$. [3]

c. The cross section of the living space under the roof can be modelled by a rectangle $CDEF$ with points $\mathrm{C}(-a,\ 0)$ and $\mathrm{D}(a,\ 0)$, where $0 < a \le 20$. [5]

Show that the maximum area A of the rectangle $CDEF$ is $600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$.

d. A function I is known as the Insulation Factor of $CDEF$. The function is defined as $I(a) = \frac{P(a)}{A(a)}$ where $\mathrm{P} = \mathrm{Perimeter}$ and $\mathrm{A} = \mathrm{Area\ of\ the\ rectangle.}$ [9]

(i) Find an expression for P in terms of a.

(ii) Find the value of a which minimizes I.

(iii) Using the value of a found in part (ii) calculate the percentage of the cross sectional area under the whole roof that is not included in the cross section of the living space.

Markscheme

a. $f'(x) = 30\mathrm{e}^{-\frac{x^2}{400}} \bullet -\frac{2x}{400} \quad \left(= -\frac{3x}{20}\mathrm{e}^{-\frac{x^2}{400}}\right)$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$$f''(x) = -\frac{3}{20}\mathrm{e}^{-\frac{x^2}{400}} + \frac{3x^2}{4000}\mathrm{e}^{-\frac{x^2}{400}} \quad \left(= \frac{3}{20}\mathrm{e}^{-\frac{x^2}{400}}\left(\frac{x^2}{200} - 1\right)\right)$$ ***M1A1***

Note: Award ***M1*** for attempting to use the product rule.

[4 marks]

b. the roof function has maximum gradient when $f''(x) = 0$ ***(M1)***

Note: Award ***(M1)*** for attempting to find $f''\left(-\sqrt{200}\right)$.

EITHER

$= 0$ ***A1***

OR

$f''(x) = 0 \Rightarrow x = \pm\sqrt{200}$ ***A1***

THEN

valid argument for maximum such as reference to an appropriate graph or change in the sign of $f''(x)$ *eg* $f''(-15) = 0.010\ldots(> 0)$ and $f''(-14) = -0.001\ldots(< 0)$ ***R1***

$\Rightarrow x = -\sqrt{200}$ ***AG***

[3 marks]

c. $A = 2a \bullet 30\mathrm{e}^{-\frac{a^2}{400}} \quad \left(= 60a\mathrm{e}^{-\frac{a^2}{400}} = -400g'(a)\right)$ ***(M1)(A1)***

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}a} = 60a\mathrm{e}^{-\frac{a^2}{400}} \bullet -\frac{a}{200} + 60\mathrm{e}^{-\frac{a^2}{400}} = 0 \Rightarrow a = \sqrt{200} \quad \left(-400f''(a) = 0 \Rightarrow a = \sqrt{200}\right)$ ***M1A1***

OR

by symmetry *eg* $a = -\sqrt{200}$ found in (b) or $A_{\max}$ coincides with $f''(a) = 0$ ***R1***

$\Rightarrow a = \sqrt{200}$ ***A1***

Note: Award ***A0(M1)(A1)M0M1*** for candidates who start with $a = \sqrt{200}$ and do not provide any justification for the maximum area. Condone use of x.

THEN

$A_{\max} = 60 \bullet \sqrt{200}\mathrm{e}^{-\frac{200}{400}}$ ***M1***

$= 600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$ ***AG***

[5 marks]

d. (i) perimeter $= 4a + 60\mathrm{e}^{-\frac{a^2}{400}}$ ***A1A1***

Note: Condone use of x.

(ii) $I(a)=\frac{4a+60\mathrm{e}^{-\frac{a^2}{400}}}{60a\mathrm{e}^{-\frac{a^2}{400}}}$ ***(A1)***

graphing $I(a)$ or other valid method to find the minimum ***(M1)***

$a=12.6$ ***A1***

(iii) area under roof $=\int_{-20}^{20}30\mathrm{e}^{-\frac{x^2}{400}}\mathrm{d}x$ ***M1***

$=896.18\ldots$ ***(A1)***

area of living space $=60\cdot(12.6...)\cdot e-\frac{(12.6...)}{400}^{2}=508.56...$

percentage of empty space $=43.3\%$ ***A1***

[9 marks]

Total [21 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Consider the curve with equation $x^3+y^3=4xy$.

The tangent to this curve is parallel to the x-axis at the point where $x=k,\ k>0$.

a. Use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4y-3x^2}{3y^2-4x}$. [3]

b. Find the value of k. [5]

Markscheme

a. $3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=4\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ ***M1A1***

$(3y^2-4x)\frac{\mathrm{d}y}{\mathrm{d}x}=4y-3x^2$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4y-3x^2}{3y^2-4x}$ ***AG***

[3 marks]

b. $\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow 4y-3x^2=0$ ***(M1)***

substituting $x=k$ and $y=\frac{3}{4}k^2$ into $x^3+y^3=4xy$ ***M1***

$k^3+\frac{27}{64}k^6=3k^3$ ***A1***

attempting to solve $k^3+\frac{27}{64}k^6=3k^3$ for k ***(M1)***

$k=1.68\ \left(=\frac{4}{3}\sqrt[3]{2}\right)$ ***A1***

Note: Condone substituting $y=\frac{3}{4}x^2$ into $x^3+y^3=4xy$ and solving for x.

[5 marks]

Examiners report

a. Part (a) was generally well done. Some use of partial differentiation accompanied by rudimentary partial derivative notation was observed in a few candidate's solutions.

b. In part (b), a large number of candidates knew to use $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and seemingly understood the required solution plan but were unable to correctly substitute $x = k$ and $y = \frac{3k^2}{4}$ into the relation and solve for k.

Consider the function $f(x) = \frac{\sqrt{x}}{\sin x}$, $0 < x < \pi$.

Consider the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$.

a.i. Show that the x-coordinate of the minimum point on the curve $y = f(x)$ satisfies the equation $\tan x = 2x$. [5]

a.ii. Determine the values of x for which $f(x)$ is a decreasing function. [2]

b. Sketch the graph of $y = f(x)$ showing clearly the minimum point and any asymptotic behaviour. [3]

c. Find the coordinates of the point on the graph of f where the normal to the graph is parallel to the line $y = -x$. [4]

d. This region is now rotated through 2π radians about the x-axis. Find the volume of revolution. [3]

Markscheme

a.i. attempt to use quotient rule or product rule ***M1***

$$f'(x) = \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x} \quad \left(= \frac{1}{2\sqrt{x}\sin x} - \frac{\sqrt{x}\cos x}{\sin^2 x}\right) \quad \textit{\textbf{A1A1}}$$

Note: Award ***A1*** for $\frac{1}{2\sqrt{x}\sin x}$ or equivalent and ***A1*** for $-\frac{\sqrt{x}\cos x}{\sin^2 x}$ or equivalent.

setting $f'(x) = 0$ ***M1***

$\frac{\sin x}{2\sqrt{x}} - \sqrt{x}\cos x = 0$

$\frac{\sin x}{2\sqrt{x}} = \sqrt{x}\cos x$ or equivalent ***A1***

$\tan x = 2x$ ***AG***

[5 marks]

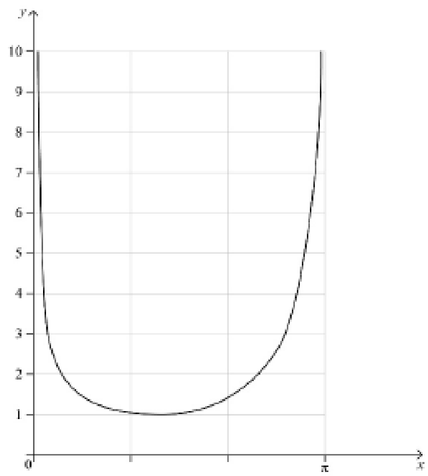
a.ii. $x = 1.17$

$0 < x \leqslant 1.17$ ***A1A1***

Note: Award ***A1*** for $0 < x$ and ***A1*** for $x \leqslant 1.17$. Accept $x < 1.17$.

[2 marks]

b.



concave up curve over correct domain with one minimum point above the x-axis. ***A1***

approaches $x = 0$ asymptotically ***A1***

approaches $x = \pi$ asymptotically ***A1***

Note: For the final ***A1*** an asymptote must be seen, and π must be seen on the x-axis or in an equation.

[3 marks]

c. $f'(x)\left(= \frac{\sin x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \sqrt{x}\cos x}{\sin^2 x}\right) = 1$ ***(A1)***

attempt to solve for x ***(M1)***

$x = 1.96$ ***A1***

$y = f(1.96\ldots)$

$= 1.51$ ***A1***

[4 marks]

d. $V = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x \mathrm{d}x}{\sin^2 x}$ ***(M1)(A1)***

Note: ***M1*** is for an integral of the correct squared function (with or without limits and/or π).

$= 2.68\ (= 0.852\pi)$ ***A1***

[3 marks]

Examiners report

a.i. [N/A]

a.ii. [N/A]

b. [N/A]

c. [N/A]

[N/A]

d.

The acceleration of a car is $\frac{1}{40}(60 - v)$ ms^{-2}, when its velocity is v ms^{-2}. Given the car starts from rest, find the velocity of the car after 30 seconds.

Markscheme

METHOD 1

$\frac{\text{d}v}{\text{d}t} = \frac{1}{40}(60 - v)$ ***(M1)***

attempting to separate variables $\int \frac{\text{d}v}{60-v} = \int \frac{\text{d}t}{40}$ ***M1***

$-\ln(60 - v) = \frac{t}{40} + c$ ***A1***

$c = -\ln 60$ (or equivalent) ***A1***

attempting to solve for v when $t = 30$ ***(M1)***

$v = 60 - 60e^{-\frac{3}{4}}$

$v = 31.7\ (\text{ms}^{-1})$ ***A1***

METHOD 2

$\frac{\text{d}v}{\text{d}t} = \frac{1}{40}(60 - v)$ ***(M1)***

$\frac{\text{d}t}{\text{d}v} = \frac{40}{60-v}$ (or equivalent) ***M1***

$\int_0^{v_f} \frac{40}{60-v}\text{d}v = 30$ where v_f is the velocity of the car after 30 seconds. ***A1A1***

attempting to solve $\int_0^{v_f} \frac{40}{60-v}\text{d}v = 30$ for v_f ***(M1)***

$v = 31.7\ (\text{ms}^{-1})$ ***A1***

[6 marks]

Examiners report

Most candidates experienced difficulties with this question. A large number of candidates did not attempt to separate the variables and instead either attempted to integrate with respect to v or employed constant acceleration formulae. Candidates that did separate the variables and attempted to integrate both sides either made a sign error, omitted the constant of integration or found an incorrect value for this constant. Almost all candidates were not aware that this question could be solved readily on a GDC.

A particle moves along a straight line so that after t seconds its displacement s , in metres, satisfies the equation $s^2 + s - 2t = 0$. Find, in terms of s , expressions for its velocity and its acceleration.

Markscheme

$2s\frac{\text{d}s}{\text{d}t} + \frac{\text{d}s}{\text{d}t} - 2 = 0$ ***M1A1***

$v = \frac{\text{d}s}{\text{d}t} = \frac{2}{2s+1}$ ***A1***

EITHER

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$ ***(M1)***
$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{-4}{(2s+1)^2}$ ***(A1)***
$a = \frac{-4}{(2s+1)^2}\frac{\mathrm{d}s}{\mathrm{d}t}$
OR
$2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + 2s\frac{\mathrm{d}^2s}{\mathrm{d}t^2} + \frac{\mathrm{d}^2s}{\mathrm{d}t^2} = 0$ ***(M1)***
$\underbrace{\frac{\mathrm{d}^2s}{\mathrm{d}t^2}}_{a} = \frac{-2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{2s+1}$ ***(A1)***
THEN
$a = \frac{-8}{(2s+1)^3}$ ***A1***
[6 marks]

Examiners report

Despite the fact that many candidates were able to calculate the speed of the particle, many of them failed to calculate the acceleration. Implicit differentiation turned out to be challenging in this exercise showing in many cases a lack of understanding of independent/dependent variables. Very often candidates did not use the chain rule or implicit differentiation when attempting to find the acceleration. It was not uncommon to see candidates trying to differentiate implicitly with respect to *t* rather than *s*, but getting the variables muddled.

Consider $f(x) = -1 + \ln\left(\sqrt{x^2-1}\right)$

The function f is defined by $f(x) = -1 + \ln\left(\sqrt{x^2-1}\right),\ x \in D$

The function g is defined by $g(x) = -1 + \ln\left(\sqrt{x^2-1}\right),\ x \in]1,\ \infty[$.

a. Find the largest possible domain D for f to be a function. [2]

b. Sketch the graph of $y = f(x)$ showing clearly the equations of asymptotes and the coordinates of any intercepts with the axes. [3]

c. Explain why f is an even function. [1]

d. Explain why the inverse function f^{-1} does not exist. [1]

e. Find the inverse function g^{-1} and state its domain. [4]

f. Find $g'(x)$. [3]

g.i. Hence, show that there are no solutions to $g'(x) = 0$; [2]

g.ii. Hence, show that there are no solutions to $(g^{-1})'(x) = 0$. [2]

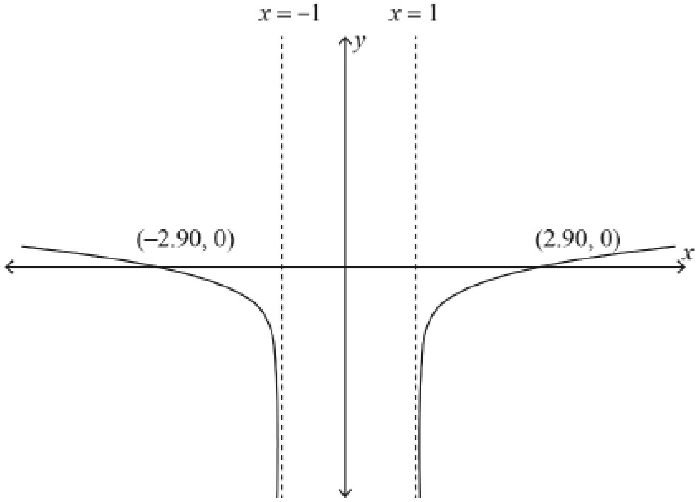
Markscheme

a. $x^2 - 1 > 0$ ***(M1)***

$x < -1$ or $x > 1$ ***A1***

[2 marks]

b.



shape ***A1***

$x = 1$ and $x = -1$ ***A1***

x-intercepts ***A1***

[3 marks]

c. **EITHER**

f is symmetrical about the y-axis ***R1***

OR

$f(-x) = f(x)$ ***R1***

[1 mark]

d. **EITHER**

f is not one-to-one function ***R1***

OR

horizontal line cuts twice ***R1***

Note: Accept any equivalent correct statement.

[1 mark]

e. $x = -1 + \ln\left(\sqrt{y^2 - 1}\right)$ ***M1***

$\mathrm{e}^{2x+2} = y^2 - 1$ ***M1***

$g^{-1}(x) = \sqrt{\mathrm{e}^{2x+2} + 1},\ x \in \mathbb{R}$ ***A1A1***

[4 marks]

f. $g'(x) = \frac{1}{\sqrt{x^2-1}} \times \frac{2x}{2\sqrt{x^2-1}}$ ***M1A1***

$g'(x) = \frac{x}{x^2-1}$ ***A1***

[3 marks]

g.i. $g'(x) = \frac{x}{x^2-1} = 0 \Rightarrow x = 0$ ***M1***

which is not in the domain of g (hence no solutions to $g'(x) = 0$) ***R1***

[2 marks]

g.ii. $(g^{-1})'(x) = \frac{e^{2x+2}}{\sqrt{e^{2x+2}+1}}$ ***M1***

as $e^{2x+2} > 0 \Rightarrow (g^{-1})'(x) > 0$ so no solutions to $(g^{-1})'(x) = 0$ ***R1***

Note: Accept: equation $e^{2x+2} = 0$ has no solutions.

[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]
g.i. [N/A]
g.ii. [N/A]

Particle A moves such that its velocity v ms^{-1}, at time t seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geqslant 0$.

Particle B moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

a. Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. [2]

b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4}\text{d}t$. [4]

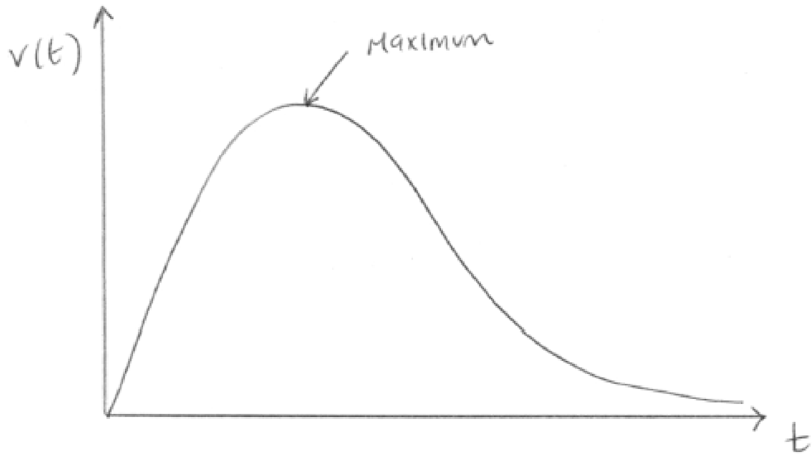
c. Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds. [3]

Give your answer in the form $k\arctan(b)$, k, $b \in \mathbb{R}$.

d. Find the acceleration of particle B when $s = 0.1$ m. [3]

Markscheme

a. (a)



A1

A1 for correct shape and correct domain

$(1.41,\ 0.0884)$ $\left(\sqrt{2},\ \frac{\sqrt{2}}{16}\right)$ ***A1***

[2 marks]

b. **EITHER**

$u = t^2$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ ***A1***

OR

$t = u^{\frac{1}{2}}$

$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$ ***A1***

THEN

$\int \frac{t}{12+t^4}\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}u}{12+u^2}$ ***M1***

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ ***M1***

$= \frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent ***A1***

[4 marks]

c. $\int_0^6 \frac{t}{12+t^4}\mathrm{d}t$ ***(M1)***

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)$ $\left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)$ (m) ***A1***

Note: Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

[3 marks]

d. $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2\sqrt{s(1-s)}}$ ***(A1)***

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$

$a = \arcsin\left(\sqrt{s}\right) \times \frac{1}{2\sqrt{s(1-s)}}$ ***(M1)***

$a = \arcsin\left(\sqrt{0.1}\right) \times \frac{1}{2\sqrt{0.1\times0.9}}$

$a = 0.536\ (\mathrm{ms}^{-2})$ ***A1***

[3 marks]

Examiners report

[N/A]

b. [N/A]
c. [N/A]
d. [N/A]

A function is defined by $f(x) = x^2 + 2,\ x \geq 0$. A region R is enclosed by $y = f(x)$,the y-axis and the line $y = 4$.

a. (i) Express the area of the region R as an integral with respect to y. [3]

(ii) Determine the area of R, giving your answer correct to four significant figures.

b. Find the exact volume generated when the region R is rotated through 2π radians about the y-axis. [3]

Markscheme

a. (i) $\text{area} = \int_2^4 \sqrt{y-2}\mathrm{d}y$ ***M1A1***

(ii) $= 1.886 \text{ (4 sf only)}$ ***A1***

Note: Award ***M0A0A1*** for finding 1.886 from $\int_0^{\sqrt{2}} 4 - f(x)\mathrm{d}x$.

Award ***A1FT*** for a 4sf answer obtained from an integral involving x.

[3 marks]

b. $\text{volume} = \pi \int_2^4 (y-2)\mathrm{d}y$ ***(M1)***

Note: Award ***M1*** for the correct integral with incorrect limits.

$= \pi\left[\frac{y^2}{2} - 2y\right]_2^4$ ***(A1)***

$= 2\pi \text{ (exact only)}$ ***A1***

[3 marks]

Total [6 marks]

Examiners report

a. [N/A]
b. [N/A]

A function f is defined by $f(x) = x^3 + \mathrm{e}^x + 1,\ x \in \mathbb{R}$. By considering $f'(x)$ determine whether f is a one-to-one or a many-to-one function.

Markscheme

$f'(x) = 3x^2 + \mathrm{e}^x$ ***A1***

Note: Accept labelled diagram showing the graph $y = f'(x)$ above the x-axis;

do not accept unlabelled graphs nor graph of $y = f(x)$.

EITHER

this is always > 0 ***R1***

so the function is (strictly) increasing ***R1***

and thus $1 - 1$ ***A1***

OR

this is always > 0 $\quad(\text{accept } \neq 0)$ ***R1***

so there are no turning points ***R1***

and thus $1 - 1$ ***A1***

Note: ***A1*** is dependent on the first ***R1***.

[4 marks]

Examiners report

The differentiation was normally completed correctly, but then a large number did not realise what was required to determine the type of the original function. Most candidates scored 1/4 and wrote explanations that showed little or no understanding of the relation between first derivative and the given function. For example, it was common to see comments about horizontal and vertical line tests but applied to the incorrect function.In term of mathematical language, it was noted that candidates used many terms incorrectly showing no knowledge of the meaning of terms like ‘parabola’, ‘even’ or ‘odd’ (or no idea about these concepts).

(a) Integrate $\int \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta$.

(b) Given that $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \frac{1}{2}$ and $\frac{\pi}{2} < a < \pi$, find the value of a .

Markscheme

(a) $\int \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \int \frac{(1-\cos\theta)'}{1-\cos\theta}\mathrm{d}\theta = \ln(1-\cos\theta) + C$ ***(M1)A1A1***

Note: Award ***A1*** for $\ln(1-\cos\theta)$ and ***A1*** for C.

(b) $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta = \frac{1}{2} \Rightarrow [\ln(1-\cos\theta)]_{\frac{\pi}{2}}^{a} = \frac{1}{2}$ ***M1***

$1 - \cos a = \mathrm{e}^{\frac{1}{2}} \Rightarrow a = \arccos\left(1-\sqrt{\mathrm{e}}\right)$ or 2.28 ***A1*** ***N2***

[5 marks]

Examiners report

Generally well answered, although many students did not include the constant of integration.

Given that the graph of $y = x^3 - 6x^2 + kx - 4$ has exactly one point at which the gradient is zero, find the value of k .

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + k$ ***M1A1***

For use of discriminant $b^2 - 4ac = 0$ or completing the square $3(x-2)^2 + k - 12$ ***(M1)***

$144 - 12k = 0$ ***(A1)***

Note: Accept trial and error, sketches of parabolas with vertex (2,0) or use of second derivative.

$k = 12$ ***A1***

[5 marks]

Examiners report

Generally candidates answer this question well using a diversity of methods. Surprisingly, a small number of candidates were successful in answering this question using the discriminant of the quadratic and in many cases reverted to trial and error to obtain the correct answer.

A ladder of length 10 m on horizontal ground rests against a vertical wall. The bottom of the ladder is moved away from the wall at a constant speed of 0.5 ms^{-1}. Calculate the speed of descent of the top of the ladder when the bottom of the ladder is 4 m away from the wall.

Markscheme

let x, y (m) denote respectively the distance of the bottom of the ladder from the wall and the distance of the top of the ladder from the ground

then,

$x^2 + y^2 = 100$ ***M1A1***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***M1A1***

when $x = 4,\ y = \sqrt{84}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5$ ***A1***

substituting, $2 \times 4 \times 0.5 + 2\sqrt{84}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}t} = -0.218\ \mathrm{ms}^{-1}$ ***A1***

(speed of descent is $0.218\ \mathrm{ms}^{-1}$)

[7 marks]

Examiners report

[N/A]

Let the function f be defined by $f(x) = \frac{2-e^x}{2e^x-1}$, $x \in D$.

a. Determine D, the largest possible domain of f. [2]

b. Show that the graph of f has three asymptotes and state their equations. [5]

c. Show that $f'(x) = -\frac{3e^x}{(2e^x-1)^2}$. [3]

d. Use your answers from parts (b) and (c) to justify that f has an inverse and state its domain. [4]

e. Find an expression for $f^{-1}(x)$. [4]

f. Consider the region R enclosed by the graph of $y = f(x)$ and the axes. [4]

Find the volume of the solid obtained when R is rotated through 2π about the y-axis.

Markscheme

a. attempting to solve either $2e^x - 1 = 0$ or $2e^x - 1 \neq 0$ for x ***(M1)***

$D = \mathbb{R} \backslash \{-\ln 2\}$ (or equivalent *eg* $x \neq -\ln 2$) ***A1***

Note: Accept $D = \mathbb{R} \backslash \{-0.693\}$ or equivalent *eg* $x \neq -0.693$.

[2 marks]

b. considering $\lim\limits_{x \to -\ln 2} f(x)$ ***(M1)***

$x = -\ln 2$ $(x = -0.693)$ ***A1***

considering one of $\lim\limits_{x \to -\infty} f(x)$ or $\lim\limits_{x \to +\infty} f(x)$ ***M1***

$\lim\limits_{x \to -\infty} f(x) = -2 \Rightarrow y = -2$ ***A1***

$\lim\limits_{x \to +\infty} f(x) = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$ ***A1***

Note: Award ***A0A0*** for $y = -2$ and $y = -\frac{1}{2}$ stated without any justification.

[5 marks]

c. $f'(x) = \frac{-e^x(2e^x-1) - 2e^x(2-e^x)}{(2e^x-1)^2}$ ***M1A1A1***

$= -\frac{3e^x}{(2e^x-1)^2}$ ***AG***

[3 marks]

d. $f'(x) < 0$ (for all $x \in D$) $\Rightarrow f$ is (strictly) decreasing ***R1***

Note: Award ***R1*** for a statement such as $f'(x) \neq 0$ and so the graph of f has no turning points.

one branch is above the upper horizontal asymptote and the other branch is below the lower horizontal asymptote ***R1***

f has an inverse ***AG***

$-\infty < x < -2 \cup -\frac{1}{2} < x < \infty$ ***A2***

Note: Award ***A2*** if the domain of the inverse is seen in either part (d) or in part (e).

[4 marks]

e. $x = \frac{2-e^y}{2e^y-1}$ ***M1***

Note: Award ***M1*** for interchanging x and y (can be done at a later stage).

$2xe^y - x = 2 - e^y$ ***M1***

$e^y(2x+1) = x+2$ ***A1***

$f^{-1}(x) = \ln\left(\frac{x+2}{2x+1}\right)$ $\left(f^{-1}(x) = \ln(x+2) - \ln(2x+1)\right)$ ***A1***

[4 marks]

f. use of $V = \pi \int_a^b x^2 dy$ ***(M1)***

$= \pi \int_0^1 \left(\ln\left(\frac{y+2}{2y+1}\right)\right)^2 dy$ ***(A1)(A1)***

Note: Award ***(A1)*** for the correct integrand and ***(A1)*** for the limits.

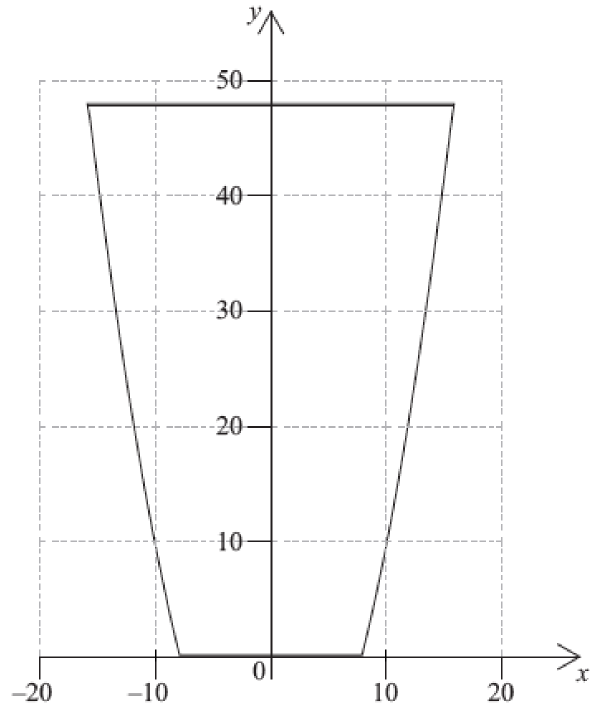
$= 0.331$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]
f. [N/A]

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is 48 cm.

a. If the container is filled with water to a depth of $h\,\text{cm}$, show that the volume, $V\,\text{cm}^3$, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. [3]

b. The container, initially full of water, begins leaking from a small hole at a rate given by \(\frac{{{\text{d}}V}}{{{\text{d}}t}} = - \frac{{250\sqrt h }}{{\pi(h + 16)}}\) where t is measured in seconds. [10]

(i) Show that \(\frac{{{\text{d}}h}}{{{\text{d}}t}} = - \frac{{250\sqrt h }}{{4{\pi ^2}{{(h + 16)}^2}}}\).

(ii) State $\frac{\text{d}t}{\text{d}h}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\text{d}h$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute)

c. Once empty, water is pumped back into the container at a rate of $8.5\ \text{cm}^3\text{s}^{-1}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\text{cm}^3\text{s}^{-1}$. [3]

Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container.

Markscheme

a. attempting to use $V = \pi\int_a^b x^2\text{d}y$ ***(M1)***

attempting to express x^2 in terms of y *ie* $x^2 = 4(y+16)$ ***(M1)***

for $y = h,\ V = 4\pi\int_0^h y + 16\text{d}y$ ***A1***

$V = 4\pi\left(\frac{h^2}{2} + 16h\right)$ ***AG***

[3 marks]

b. (i) **METHOD 1**

$\frac{\text{d}h}{\text{d}t} = \frac{\text{d}h}{\text{d}V} \times \frac{\text{d}V}{\text{d}t}$ ***(M1)***

$\frac{\text{d}V}{\text{d}h} = 4\pi(h+16)$ ***(A1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$ ***M1A1***

Note: Award ***M1*** for substitution into $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{250\sqrt{h}}{4\pi^2(h+16)^2}$ ***AG***

METHOD 2

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (implicit differentiation)***(M1)***

$\frac{-250\sqrt{h}}{\pi(h+16)} = 4\pi(h+16)\frac{\mathrm{d}h}{\mathrm{d}t}$ (or equivalent) ***A1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi(h+16)} \times \frac{-250\sqrt{h}}{\pi(h+16)}$ ***M1A1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{250\sqrt{h}}{4\pi^2(h+16)^2}$ ***AG***

(ii) $\frac{\mathrm{d}t}{\mathrm{d}h} = -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}$ ***A1***

$t = \int -\frac{4\pi^2(h+16)^2}{250\sqrt{h}}\mathrm{d}h$ ***(M1)***

$t = \int -\frac{4\pi^2(h^2+32h+256)}{250\sqrt{h}}\mathrm{d}h$ ***A1***

$t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$ ***AG***

(iii) **METHOD 1**

$t = \frac{-4\pi^2}{250}\int_{48}^{0}\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$ ***(M1)***

$t = 2688.756\ldots\ (\mathrm{s})$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

METHOD 2

$t = \frac{-4\pi^2}{250}\left(\frac{2}{5}h^{\frac{5}{2}} + \frac{64}{3}h^{\frac{3}{2}} + 512h^{\frac{1}{2}}\right) + c$

when $t = 0,\ h = 48 \Rightarrow c = 2688.756\ldots\left(c = \frac{4\pi^2}{250}\left(\frac{2}{5} \times 48^{\frac{5}{2}} + \frac{64}{3} \times 48^{\frac{3}{2}} + 512 \times 48^{\frac{1}{2}}\right)\right)$ ***(M1)***

when $h = 0,\ t = 2688.756\ldots\left(t = \frac{4\pi^2}{250}\left(\frac{2}{5} \times 48^{\frac{5}{2}} + \frac{64}{3} \times 48^{\frac{3}{2}} + 512 \times 48^{\frac{1}{2}}\right)\right)\ (\mathrm{s})$ ***(A1)***

45 minutes (correct to the nearest minute) ***A1***

[10 marks]

c. **EITHER**

the depth stabilizes when $\frac{\mathrm{d}V}{\mathrm{d}t} = 0\ \ ie\ \ 8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ ***R1***

attempting to solve $8.5 - \frac{250\sqrt{h}}{\pi(h+16)} = 0$ for h ***(M1)***

OR

the depth stabilizes when $\frac{\mathrm{d}h}{\mathrm{d}t} = 0\ \ ie\ \ \frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ ***R1***

attempting to solve $\frac{1}{4\pi(h+16)}\left(8.5 - \frac{250\sqrt{h}}{\pi(h+16)}\right) = 0$ for h ***(M1)***

THEN

$h = 5.06\ (\mathrm{cm})$ ***A1***

[3 marks]

Total [16 marks]

Examiners report

a. This question was done reasonably well by a large proportion of candidates. Many candidates however were unable to show the required result in part (a). A number of candidates seemingly did not realize how the container was formed while other candidates attempted to fudge the result.

b. Part (b) was quite well done. In part (b) (i), most candidates were able to correctly calculate $\frac{\mathrm{d}V}{\mathrm{d}h}$ and correctly apply a related rates expression to show the given result. Some candidates however made a sign error when stating $\frac{\mathrm{d}V}{\mathrm{d}t}$. A large number of candidates successfully answered part (b) (ii). In part (b) (iii), successful candidates either set up and calculated an appropriate definite integral or antidifferentiated and found that $t = C$ when $h = 0$.

c. In part (c), a pleasing number of candidates realized that the water depth stabilized when either $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ or $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, sketched an appropriate graph and found the correct value of h. Some candidates misinterpreted the situation and attempted to find the coordinates of the local minimum of their graph.

Let $f(x) = \frac{\mathrm{e}^{2x}+1}{\mathrm{e}^x-2}$.

The line L_2 is parallel to L_1 and tangent to the curve $y = f(x)$.

a. Find the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. [4]

b. (i) Find $f'(x)$. [8]

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point.

c. Find the equation of L_1, the normal to the curve at the point where it crosses the y-axis. [4]

d. Find the equation of the line L_2. [5]

Markscheme

a. $x \to -\infty \Rightarrow y \to -\frac{1}{2}$ so $y = -\frac{1}{2}$ is an asymptote ***(M1)A1***

$\mathrm{e}^x - 2 = 0 \Rightarrow x = \ln 2$ so $x = \ln 2\ (= 0.693)$ is an asymptote ***(M1)A1***

[4 marks]

b. (i) $f'(x) = \frac{2(\mathrm{e}^x-2)\mathrm{e}^{2x}-(\mathrm{e}^{2x}+1)\mathrm{e}^x}{(\mathrm{e}^x-2)^2}$ ***M1A1***

$= \frac{\mathrm{e}^{3x}-4\mathrm{e}^{2x}-\mathrm{e}^x}{(\mathrm{e}^x-2)^2}$

(ii) $f'(x) = 0$ when $\mathrm{e}^{3x} - 4\mathrm{e}^{2x} - \mathrm{e}^x = 0$ ***M1***

$\mathrm{e}^x\left(\mathrm{e}^{2x} - 4\mathrm{e}^x - 1\right) = 0$

$\mathrm{e}^x = 0,\ \mathrm{e}^x = -0.236,\ \mathrm{e}^x = 4.24$ (or $\mathrm{e}^x = 2 \pm \sqrt{5}$) ***A1A1***

Note: Award ***A1*** for zero, ***A1*** for other two solutions.
Accept any answers which show a zero, a negative and a positive.

as $e^x > 0$ exactly one solution ***R1***

Note: Do not award marks for purely graphical solution.

(iii) (1.44, 8.47) ***A1A1***

[8 marks]

c. $f'(0) = -4$ ***(A1)***

so gradient of normal is $\frac{1}{4}$ ***(M1)***

$f(0) = -2$ ***(A1)***

so equation of L_1 is $y = \frac{1}{4}x - 2$ ***A1***

[4 marks]

d. $f'(x) = \frac{1}{4}$ ***M1***

so $x = 1.46$ ***(M1)A1***

$f(1.46) = 8.47$ ***(A1)***

equation of L_2 is $y - 8.47 = \frac{1}{4}(x - 1.46)$ ***A1***

(or $y = \frac{1}{4}x + 8.11$)

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

The region A is enclosed by the graph of $y = 2\arcsin(x-1) - \frac{\pi}{4}$, the y-axis and the line $y = \frac{\pi}{4}$.

a. Write down a definite integral to represent the area of A. [4]

b. Calculate the area of A. [2]

Markscheme

a. **METHOD 1**

$2\arcsin(x-1) - \frac{\pi}{4} = \frac{\pi}{4}$ ***(M1)***

$x = 1 + \frac{1}{\sqrt{2}}$ $(= 1.707\ldots)$ ***(A1)***

$\int_0^{1+\frac{1}{\sqrt{2}}} \frac{\pi}{4} - \left(2\arcsin(x-1) - \frac{\pi}{4}\right) dx$ ***M1A1***

Note: Award ***M1*** for an attempt to find the difference between two functions, ***A1*** for all correct.

METHOD 2

when $x = 0,\ y = \frac{-5\pi}{4}\ \ (= -3.93)$ ***A1***

$x = 1 + \sin\left(\frac{4y+\pi}{8}\right)$ ***M1A1***

Note: Award ***M1*** for an attempt to find the inverse function.

$\int_{\frac{-5\pi}{4}}^{\frac{\pi}{4}} \left(1 + \sin\left(\frac{4y+\pi}{8}\right)\right) \mathrm{d}y$ ***A1***

METHOD 3

$\int_0^{1.38\ldots} \left(2\arcsin(x-1) - \frac{\pi}{4}\right) \mathrm{d}x\ | + \int\limits_0^{1.71\ldots} \frac{\pi}{4} \mathrm{d}x - \int\limits_{1.38\ldots}^{1.71\ldots} \left(2\arcsin(x-1) - \frac{\pi}{4}\right) dx$ ***M1A1A1A1***

Note: Award ***M1*** for considering the area below the x-axis and above the x-axis and ***A1*** for each correct integral.

[4 marks]

b. $\text{area} = 3.30\ (\text{square units})$ ***A2***

[2 marks]

Examiners report

a. [N/A]
b. [N/A]

The displacement, s, in metres, of a particle t seconds after it passes through the origin is given by the expression $s = \ln(2 - e^{-t}),\ t \geqslant 0$.

a. Find an expression for the velocity, v, of the particle at time t. [2]

b. Find an expression for the acceleration, a, of the particle at time t. [2]

c. Find the acceleration of the particle at time $t = 0$. [1]

Markscheme

a. $v - \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{e}^{-t}}{2-\mathrm{e}^{-t}}\ \left(= \frac{1}{2\mathrm{e}^{t}-1} \text{ or } -1 + \frac{2}{2-\mathrm{e}^{-t}}\right)$ ***M1A1***

[2 marks]

b. $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{-\mathrm{e}^{-t}(2-\mathrm{e}^{-t}) - \mathrm{e}^{-t} \times \mathrm{e}^{-t}}{(2-\mathrm{e}^{-t})^2}\ \left(= \frac{-2\mathrm{e}^{-t}}{(2-\mathrm{e}^{-t})^2}\right)$ ***M1A1***

Note: If simplified in part (a) award ***(M1)A1*** for $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{-2\mathrm{e}^t}{(2\mathrm{e}^t - 1)^2}$.

Note: Award ***M1A1*** for $a = -\mathrm{e}^{-t}(2 - \mathrm{e}^{-t})^{-2}(\mathrm{e}^{-t}) - \mathrm{e}^{-t}(2 - \mathrm{e}^{-t})^{-1}$.

[2 marks]

c. $a = -2\ (\mathrm{ms}^{-2})$ ***A1***

[1 mark]

Examiners report

a. Mostly well done. There were a few sign errors but most candidates were correctly applying the quotient or chain rules.

b. Mostly well done. There were a few sign errors but most candidates were correctly applying the quotient or chain rules.

c. Mostly well done. There were a few sign errors but most candidates were correctly applying the quotient or chain rules.

(a) Differentiate $f(x) = \arcsin x + 2\sqrt{1 - x^2}$, $x \in [-1, 1]$.

(b) Find the coordinates of the point on the graph of $y = f(x)$ in $[-1, 1]$, where the gradient of the tangent to the curve is zero.

Markscheme

(a) $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \quad \left(= \frac{1-2x}{\sqrt{1-x^2}}\right)$ ***M1A1A1***

Note: Award ***A1*** for first term,
M1A1 for second term (***M1*** for attempting chain rule).

(b) $f'(x) = 0$ ***(M1)***

$x = 0.5$, $y = 2.26$ or $\frac{\pi}{6} + \sqrt{3}$ (accept (0.500, 2.26) ***A1A1*** ***N3***

[6 marks]

Examiners report

Most candidates scored well on this question, showing competence at non-trivial differentiation. The follow through rules allowed candidates to recover from minor errors in part (a). Some candidates demonstrated their resourcefulness in using their GDC to answer part (b) even when they had been unable to gain full marks on part (a).

Find the volume of the solid formed when the region bounded by the graph of $y = \sin(x - 1)$, and the lines $y = 0$ and $y = 1$ is rotated by 2π about the y-axis.

Markscheme

volume $= \pi \int x^2 \mathrm{d}y$ *(M1)*

$x = \arcsin y + 1$ *(M1)(A1)*

volume $= \pi \int_0^1 (\arcsin y + 1)^2 \mathrm{d}y$ *A1*

Note: *A1* is for the limits, provided a correct integration of y.

$= 2.608993\ldots\pi = 8.20$ *A2* *N5*

[6 marks]

Examiners report

Although it was recognised that the imprecise nature of the wording of the question caused some difficulties, these were overwhelmingly by candidates who were attempting to rotate around the x-axis. The majority of students who understood to rotate about the y-axis had no difficulties in writing the correct integral. Marks lost were for inability to find the correct value of the integral on the GDC (some clearly had the calculator in degrees) and also for poor rounding where the GDC had been used correctly. In the few instances where students seemed confused by the lack of precision in the question, benefit of the doubt was given and full points awarded.

By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y} \mathrm{d}y,\ y > 0\,.$$

Markscheme

Let $u = \ln y \Rightarrow \mathrm{d}u = \frac{1}{y}\mathrm{d}y$ *A1(A1)*

$\int \frac{\tan(\ln y)}{y} \mathrm{d}y = \int \tan u \mathrm{d}u$ *A1*

$= \int \frac{\sin u}{\cos u} \mathrm{d}u = -\ln|\cos u| + c$ *A1*

EITHER

$\int \frac{\tan(\ln y)}{y} \mathrm{d}y = -\ln|\cos(\ln y)| + c$ *A1A1*

OR

$\int \frac{\tan(\ln y)}{y} \mathrm{d}y = \ln|\sec(\ln y)| + c$ *A1A1*

[6 marks]

Examiners report

Many candidates obtained the first three marks, but then attempted various methods unsuccessfully. Quite a few candidates attempted integration by parts rather than substitution. The candidates who successfully integrated the expression often failed to put the absolute value sign in the final answer.

A stalactite has the shape of a circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Determine if its volume is increasing or decreasing, and the rate at which the volume is changing.

Markscheme

$V = \frac{\pi}{3} r^2 h$

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}\left[2rh\frac{\mathrm{d}r}{\mathrm{d}t} + r^2\frac{\mathrm{d}h}{\mathrm{d}t}\right]$ ***M1A1A1***

at the given instant

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}\left[2(4)(200)\left(-\frac{1}{2}\right) + 40^2(3)\right]$ ***M1***

$= \frac{-3200\pi}{3} = -3351.03\ldots \approx 3350$ ***A1***

hence, the volume is decreasing (at approximately 3350 mm^3 per century) ***R1***

[6 marks]

Examiners report

Few candidates applied the method of implicit differentiation and related rates correctly. Some candidates incorrectly interpreted this question as one of constant linear rates.

Consider the triangle PQR where $\mathrm{Q\hat{P}R} = 30°$, $\mathrm{PQ} = (x+2)\ \mathrm{cm}$ and $\mathrm{PR} = (5-x)^2\ \mathrm{cm}$, where $-2 < x < 5$.

a. Show that the area, $A\ \mathrm{cm}^2$, of the triangle is given by $A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$. [2]

b. (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$.

c. (i) Find $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR. [7]

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum.

Markscheme

a. use of $A = \frac{1}{2}qr\sin\theta$ to obtain $A = \frac{1}{2}(x+2)(5-x)^2\sin 30°$ ***M1***

$= \frac{1}{4}(x+2)(25-10x+x^2)$ ***A1***

$A = \frac{1}{4}(x^3-8x^2+5x+50)$ ***AG***

[2 marks]

b. (i) $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{4}(3x^2-16x+5) = \frac{1}{4}(3x-1)(x-5)$ ***A1***

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{4}\left(3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5\right) = 0$ ***M1A1***

OR

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{1}{4}\left(3\left(\frac{1}{3}\right) - 1\right)\left(\left(\frac{1}{3}\right) - 5\right) = 0$ ***M1A1***

THEN

so $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***AG***

METHOD 2

solving $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ for x ***M1***

$-2 < x < 5 \Rightarrow x = \frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***AG***

METHOD 3

a correct graph of $\frac{\mathrm{d}A}{\mathrm{d}x}$ versus x ***M1***

the graph clearly showing that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***A1***

so $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$ ***AG***

[3 marks]

c. (i) $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{1}{2}(3x-8)$ ***A1***

for $x = \frac{1}{3}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -3.5\ (<0)$ ***R1***

so $x = \frac{1}{3}$ gives the maximum area of triangle PQR ***AG***

(ii) $A_{\max} = \frac{343}{27}\ (= 12.7)\ (\mathrm{cm}^2)$ ***A1***

(iii) $\mathrm{PQ} = \frac{7}{3}\ (\mathrm{cm})$ and $\mathrm{PR} = \left(\frac{14}{3}\right)^2\ (\mathrm{cm})$ ***(A1)***

$\mathrm{QR}^2 = \left(\frac{7}{3}\right)^2 + \left(\frac{14}{3}\right)^4 - 2\left(\frac{7}{3}\right)\left(\frac{14}{3}\right)^2 \cos 30^\circ$ ***(M1)(A1)***

$= 391.702\ldots$

$\mathrm{QR} = 19.8\ (\mathrm{cm})$ ***A1***

[7 marks]

Total [12 marks]

Examiners report

a. This question was generally well done. Parts (a) and (b) were straightforward and well answered.

b. This question was generally well done. Parts (a) and (b) were straightforward and well answered.

c. This question was generally well done. Parts (c) (i) and (ii) were also well answered with most candidates correctly applying the second derivative test and displaying sound reasoning skills.

Part (c) (iii) required the use of the cosine rule and was reasonably well done. The most common error committed by candidates in attempting to find the value of QR was to use $\mathrm{PR} = \frac{14}{3}\ (\mathrm{cm})$ rather than $\mathrm{PR} = \left(\frac{14}{3}\right)^2\ (\mathrm{cm})$. The occasional candidate used $\cos 30^\circ = \frac{1}{2}$.

The particle P moves along the x-axis such that its velocity, $v\ \mathrm{ms}^{-1}$, at time t seconds is given by $v = \cos(t^2)$.

a. Given that P is at the origin O at time $t = 0$, calculate [4]

(i) the displacement of P from O after 3 seconds;

(ii) the total distance travelled by P in the first 3 seconds.

b. Find the time at which the total distance travelled by P is 1 m. [2]

Markscheme

a. (i) displacement $= \int_0^3 v\mathrm{d}t$ ***(M1)***

$= 0.703\ (\mathrm{m})$ ***A1***

(ii) total distance $= \int_0^3 |v|\,\mathrm{d}t$ ***(M1)***

$= 2.05\ (\mathrm{m})$ ***A1***

[4 marks]

b. solving the equation $\int_0^t \left|\cos(u^2)\right| \mathrm{d}u = 1$ ***(M1)***

$t = 1.39\ (\mathrm{s})$ ***A1***

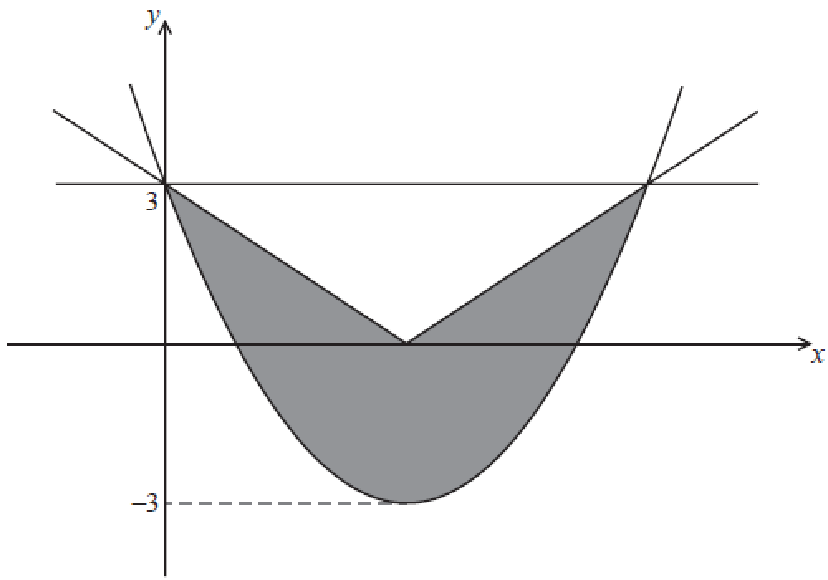
[2 marks]

Examiners report

a. [N/A]

b. [N/A]

The diagram below shows the graphs of $y = \left|\frac{3}{2}x - 3\right|,\ y = 3$ and a quadratic function, that all intersect in the same two points.



Given that the minimum value of the quadratic function is −3, find an expression for the area of the shaded region in the form $\int_0^t (ax^2 + bx + c)\mathrm{d}x$, where the constants a, b, c and t are to be determined. (Note: The integral does not need to be evaluated.)

Markscheme

$\left|\frac{3}{2}x - 3\right| = 0$ when $x = 2$ ***(A1)***

the equation of the parabola is $y = p(x-2)^2 - 3$ ***(M1)***

through $(0,\ 3) \Rightarrow 3 = 4p - 3 \Rightarrow p = \frac{3}{2}$ ***(M1)***

the equation of the parabola is $y = \frac{3}{2}(x-2)^2 - 3\ \left(= \frac{3}{2}x^2 - 6x + 3\right)$ ***A1***

area $= 2\int_0^2 \left(3 - \frac{3}{2}x\right) - \left(\frac{3}{2}x^2 - 6x + 3\right)\mathrm{d}x$ ***M1M1A1***

Note: Award ***M1*** for recognizing symmetry to obtain $2\int_0^2$,

M1 for the difference,

A1 for getting all parts correct.

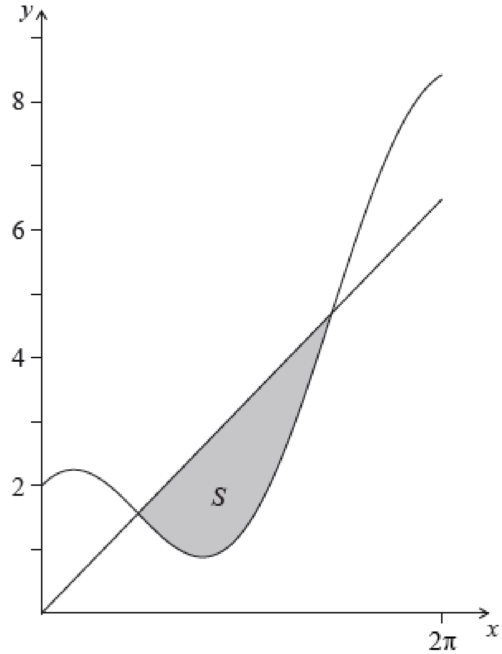
$= \int_0^2 (-3x^2 + 9x)\mathrm{d}x$ ***A1***

[8 marks]

Examiners report

This was a difficult question and, although many students obtained partial marks, there were few completely correct solutions.

The shaded region S is enclosed between the curve $y = x + 2\cos x$, for $0 \leqslant x \leqslant 2\pi$, and the line $y = x$, as shown in the diagram below.



a. Find the coordinates of the points where the line meets the curve. [3]

b. The region S is rotated by 2π about the x-axis to generate a solid. [5]

(i) Write down an integral that represents the volume V of the solid.

(ii) Find the volume V.

Markscheme

a. (a) $\frac{\pi}{2}(1.57),\ \frac{3\pi}{2}(4.71)$ ***A1A1***

hence the coordinates are $\left(\frac{\pi}{2},\ \frac{\pi}{2}\right),\ \left(\frac{3\pi}{2},\ \frac{3\pi}{2}\right)$ ***A1***

[3 marks]

b. (i) $\pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(x^2-(x+2\cos x)^2\right)\mathrm{d}x$ ***A1A1A1***

Note: Award ***A1*** for $x^2-(x+2\cos x)^2$, ***A1*** for correct limits and ***A1*** for π.

(ii) $6\pi^2\ (=59.2)$ ***A2***

Notes: Do not award **ft** from (b)(i).

[5 marks]

Examiners report

a. [N/A]

b. [N/A]

Consider the curve, C defined by the equation $y^2-2xy=5-\mathrm{e}^x$. The point A lies on C and has coordinates $(0,\ a),\ a>0$.

a. Find the value of a. [2]

b. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y-\mathrm{e}^x}{2(y-x)}$. [4]

c. Find the equation of the normal to C at the point A. [3]

d. Find the coordinates of the second point at which the normal found in part (c) intersects C. [4]

e. Given that $v = y^3,\ y > 0$, find $\frac{\mathrm{d}v}{\mathrm{d}x}$ at $x = 0$. [3]

Markscheme

a. $a^2 = 5 - 1$ ***(M1)***

$a = 2$ ***A1***

[2 marks]

b. $2y\frac{\mathrm{d}y}{\mathrm{d}x} - \left(2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\right) = -\mathrm{e}^x$ ***M1A1A1A1***

Note: Award ***M1*** for an attempt at implicit differentiation, ***A1*** for each part.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y-\mathrm{e}^x}{2(y-x)}$ ***AG***

[4 marks]

c. at $x = 0,\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{4}$ ***(A1)***

finding the negative reciprocal of a number ***(M1)***

gradient of normal is $-\frac{4}{3}$

$y = -\frac{4}{3}x + 2$ ***A1***

[3 marks]

d. substituting linear expression ***(M1)***

$\left(-\frac{4}{3}x + 2\right)^2 - 2x\left(-\frac{4}{3}x + 2\right) + \mathrm{e}^x - 5 = 0$ or equivalent

$x = 1.56$ ***(M1)A1***

$y = -0.0779$ ***A1***

$(1.56,\ -0.0779)$

[4 marks]

e. $\frac{\mathrm{d}v}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = 3 \times 4 \times \frac{3}{4} = 9$ ***A1***

[3 marks]

Examiners report

a. Parts (a) to (c) were generally well done.

b. Parts (a) to (c) were generally well done.

c. Parts (a) to (c) were generally well done although a significant number of students found the equation of the tangent rather than the normal in part (c).

d. Whilst many were able to make a start on part (d), fewer students had the necessary calculator skills to work it though correctly.

e. There were many overly complicated solutions to part (e), some of which were successful.

If $y = \ln\left(\frac{1}{3}(1+\mathrm{e}^{-2x})\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}(\mathrm{e}^{-y} - 3)$.

Markscheme

$y = \ln\left(\frac{1}{3}(1+\mathrm{e}^{-2x})\right)$

EITHER

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{2}{3}\mathrm{e}^{-2x}}{\frac{1}{3}(1+\mathrm{e}^{-2x})}$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\mathrm{e}^{-2x}}{1+\mathrm{e}^{-2x}}$ ***A1***

$\mathrm{e}^y = \frac{1}{3}(1+\mathrm{e}^{-2x})$ ***M1***

Now $\mathrm{e}^{-2x} = 3\mathrm{e}^y - 1$ ***A1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2(3\mathrm{e}^y-1)}{1+3\mathrm{e}^y-1}$ ***A1***

$= -\frac{2}{3\mathrm{e}^y}(3\mathrm{e}^y - 1)$

$= -\frac{2}{3}(3 - \mathrm{e}^{-y})$ ***A1***

$= \frac{2}{3}(\mathrm{e}^{-y} - 3)$ ***AG***

OR

$\mathrm{e}^y = \frac{1}{3}(1+\mathrm{e}^{-2x})$ ***M1A1***

$\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}\mathrm{e}^{-2x}$ ***M1A1***

Now $\mathrm{e}^{-2x} = 3\mathrm{e}^y - 1$ ***(A1)***

$\Rightarrow \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}(3\mathrm{e}^y - 1)$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3}\mathrm{e}^{-y}(3\mathrm{e}^y - 1)$ ***(A1)***

$= \frac{2}{3}(-3 + \mathrm{e}^{-y})$ ***(A1)***

$= \frac{2}{3}(\mathrm{e}^{-y} - 3)$ **AG**

Note: Only two of the three ***(A1)*** marks may be implied.

[7 marks]

Examiners report

Solutions were generally disappointing with many candidates being awarded the first 2 or 3 marks, but then going no further.

A point P moves in a straight line with velocity $v$ms^{-1} given by $v\left(t\right) = \mathrm{e}^{-t} - 8t^2\mathrm{e}^{-2t}$ at time t seconds, where $t \geq 0$.

a. Determine the first time t_1 at which P has zero velocity. [2]

b.i. Find an expression for the acceleration of P at time t. [2]

b.ii. Find the value of the acceleration of P at time t_1. [1]

Markscheme

a. attempt to solve $v\left(t\right) = 0$ for t or equivalent ***(M1)***

t_1 = 0.441(s) ***A1***

[2 marks]

b.i. $a\left(t\right) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{e}^{-t} - 16t\mathrm{e}^{-2t} + 16t^2\mathrm{e}^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

[2 marks]

b.ii. $a\left(t_1\right) = -2.28$(ms^{-2}) ***A1***

[1 mark]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

Let $f(x) = x(x+2)^6$.

a. Solve the inequality $f(x) > x$. [5]

b. Find $\int f(x)\mathrm{d}x$. [5]

Markscheme

a. **METHOD 1**

sketch showing where the lines cross or zeros of $y = x(x+2)^6 - x$ ***(M1)***
$x = 0$ ***(A1)***
$x = -1$ and $x = -3$ ***(A1)***
the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

METHOD 2

separating into two cases $x > 0$ and $x < 0$ ***(M1)***
if $x > 0$ then $(x+2)^6 > 1 \Rightarrow$ always true ***(M1)***
if $x < 0$ then $(x+2)^6 < 1 \Rightarrow -3 < x < -1$ ***(M1)***
so the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

METHOD 3
$f(x) = x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x$ ***(A1)***
solutions to $x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 63x = 0$ are ***(M1)***
$x = 0,\ x = -1$ and $x = -3$ ***(A1)***
so the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

METHOD 4
$f(x) = x$ when $x(x+2)^6 = x$
either $x = 0$ or $(x+2)^6 = 1$ ***(A1)***
if $(x+2)^6 = 1$ then $x + 2 = \pm 1$ so $x = -1$ or $x = -3$ ***(M1)(A1)***
the solution is $-3 < x < -1$ or $x > 0$ ***A1A1***

Note: Do not award either final ***A1*** mark if strict inequalities are not given.

[5 marks]

b. **METHOD 1** (by substitution)

substituting $u = x + 2$ ***(M1)***
$\mathrm{d}u = \mathrm{d}x$
$\int (u-2)u^6 \mathrm{d}u$ ***M1A1***
$= \frac{1}{8}u^8 - \frac{2}{7}u^7 (+c)$ ***(A1)***
$= \frac{1}{8}(x+2)^8 - \frac{2}{7}(x+2)^7 (+c)$ ***A1***
METHOD 2 (by parts)
$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1,\ \frac{\mathrm{d}v}{\mathrm{d}x} = (x+2)^6 \Rightarrow v = \frac{1}{7}(x+2)^7$ ***(M1)(A1)***
$\int x(x+2)^6 \mathrm{d}x = \frac{1}{7}x(x+2)^7 - \frac{1}{7}\int (x+2)^7 \mathrm{d}x$ ***M1***
$= \frac{1}{7}x(x+2)^7 - \frac{1}{56}(x+2)^8 (+c)$ ***A1A1***
METHOD 3 (by expansion)
$\int f(x)\mathrm{d}x = \int \left(x^7 + 12x^6 + 60x^5 + 160x^4 + 240x^3 + 192x^2 + 64x\right) \mathrm{d}x$ ***M1A1***
$= \frac{1}{8}x^8 + \frac{12}{7}x^7 + 10x^6 + 32x^5 + 60x^4 + 64x^3 + 32x^2 (+c)$ ***M1A2***

Note: Award ***M1A1*** if at least four terms are correct.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the function f , defined by $f(x) = x - a\sqrt{x}$, where $x \geqslant 0$, $a \in \mathbb{R}^+$.

(a) Find in terms of a

(i) the zeros of f ;

(ii) the values of x for which f is decreasing;

(iii) the values of x for which f is increasing;

(iv) the range of f .

(b) State the concavity of the graph of f .

Markscheme

(a)

(i) $x - a\sqrt{x}$ ***M1***

$\sqrt{x}\sqrt{x} - a = 0$ ***(A1)***

$2\ x = 0$, $x = a^2$ ***A1*** ***N2***

(ii) $f'(x) = 1 - \frac{a}{2\sqrt{x}}$ ***A1***

f is decreasing when $f' < 0$ ***(M1)***

$1 - \frac{a}{2\sqrt{x}} < 0 \Rightarrow \frac{2\sqrt{x}-a}{2\sqrt{x}} < 0 \Rightarrow x > \frac{a^2}{4}$ ***A1***

(iii) f is increasing when $f' > 0$

$1 - \frac{a}{2\sqrt{x}} > 0 \Rightarrow \frac{2\sqrt{x}-a}{2\sqrt{x}} > 0 \Rightarrow x > \frac{a^2}{4}$ ***A1***

Note: Award the ***M1*** mark for either (ii) or (iii).

(iv) minimum occurs at $x = \frac{a^2}{4}$

minimum value is $y = -\frac{a^2}{4}$ ***(M1)A1***

hence $y \geqslant -\frac{a^2}{4}$ ***A1***

[10 marks]

(b) concave up for all values of x ***R1***

[1 mark]

Total [11 marks]

Examiners report

This was generally a well answered question.

By using the substitution $x^2 = 2\sec\theta$, show that $\int \frac{\mathrm{d}x}{x\sqrt{x^4-4}} = \frac{1}{4}\text{arccos}\left(\frac{2}{x^2}\right) + c$.

Markscheme

EITHER

$x^2 = 2\sec\theta$

$2x\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec\theta\tan\theta$ ***M1A1***

$\int \frac{\mathrm{d}x}{x\sqrt{x^4-4}}$

$= \int \frac{\sec\theta\tan\theta\mathrm{d}\theta}{2\sec\theta\sqrt{4\sec^2\theta-4}}$ ***M1A1***

OR

$x = \sqrt{2}(\sec\theta)^{\frac{1}{2}}\ \left(= \sqrt{2}(\cos\theta)^{-\frac{1}{2}}\right)$

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\sqrt{2}}{2}(\sec\theta)^{\frac{1}{2}}\tan\theta\ \left(= \frac{\sqrt{2}}{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta\right)$ ***M1A1***

$\int \frac{\mathrm{d}x}{x\sqrt{x^4-4}}$

$= \int \frac{\sqrt{2}(\sec\theta)^{\frac{1}{2}}\tan\theta\mathrm{d}\theta}{2\sqrt{2}(\sec\theta)^{\frac{1}{2}}\sqrt{4\sec^2\theta-4}}\ \left(= \int \frac{\sqrt{2}(\cos\theta)^{-\frac{3}{2}}\sin\theta\mathrm{d}\theta}{2\sqrt{2}(\cos\theta)^{-\frac{1}{2}}\sqrt{4\sec^2\theta-4}}\right)$ ***M1A1***

THEN

$= \frac{1}{2}\int \frac{\tan\theta\mathrm{d}\theta}{2\tan\theta}$ ***(M1)***

$= \frac{1}{4}\int \mathrm{d}\theta$

$= \frac{\theta}{4} + c$ ***A1***

$x^2 = 2\sec\theta \Rightarrow \cos\theta = \frac{2}{x^2}$ ***M1***

Note: This ***M1*** may be seen anywhere, including a sketch of an appropriate triangle.

so $\frac{\theta}{4} + c = \frac{1}{4}\arccos\left(\frac{2}{x^2}\right) + c$ ***AG***

[7 marks]

Examiners report

[N/A]

The graph of $y = \ln(5x+10)$ is obtained from the graph of $y = \ln x$ by a translation of a units in the direction of the x-axis followed by a translation of b units in the direction of the y-axis.

a. Find the value of a and the value of b. [4]

b. The region bounded by the graph of $y = \ln(5x+10)$, the x-axis and the lines $x = \mathrm{e}$ and $x = 2\mathrm{e}$, is rotated through 2π radians about the x-axis. Find the volume generated. [2]

Markscheme

a. **EITHER**

$y = \ln(x - a) + b = \ln(5x + 10)$ ***(M1)***

$y = \ln(x - a) + \ln c = \ln(5x + 10)$

$y = \ln(c(x - a)) = \ln(5x + 10)$ ***(M1)***

OR

$y = \ln(5x + 10) = \ln(5(x + 2))$ ***(M1)***

$y = \ln(5) + \ln(x + 2)$ ***(M1)***

THEN

$a = -2,\ b = \ln 5$ ***A1A1***

Note: Accept graphical approaches.

Note: Accept $a = 2,\ b = 1.61$

[4 marks]

b. $V = \pi \int_e^{2e} [\ln(5x + 10)]^2 \mathrm{d}x$ ***(M1)***

$= 99.2$ ***A1***

[2 marks]

Total [6 marks]

Examiners report

a. [N/A]
b. [N/A]

Two cyclists are at the same road intersection. One cyclist travels north at $20\,\mathrm{km\,h^{-1}}$. The other cyclist travels west at $15\,\mathrm{km\,h^{-1}}$.

Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

Markscheme

METHOD 1

attempt to set up (diagram, vectors) ***(M1)***

correct distances $x = 15t,\ y = 20t$ ***(A1) (A1)***

the distance between the two cyclists at time t is $s = \sqrt{(15t)^2 + (20t)^2} = 25t\ (\mathrm{km})$ ***A1***

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$ ***A1***

hence the rate is independent of time ***AG***

METHOD 2

attempting to differentiate $x^2 + y^2 = s^2$ implicitly ***(M1)***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2s\frac{\mathrm{d}s}{\mathrm{d}t}$ ***(A1)***

the distance between the two cyclists at time t is $\sqrt{(15t)^2 + (20t)^2} = 25t\ (\mathrm{km})$ ***(A1)***

$2(15t)(15 + 2(20t)(20) = 2(25t)\frac{\mathrm{d}s}{\mathrm{d}t}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their equation involving $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$ ***A1***

hence the rate is independent of time ***AG***

METHOD 3

$s = \sqrt{x^2 + y^2}$ ***(A1)***

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}}{\sqrt{x^2+y^2}}$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to differentiate the expression for s.

$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{(15t)(15)+(20t)(20)}{\sqrt{(15t)^2+(20t)^2}}$ ***M1***

Note: Award ***M1*** for substitution of correct values into their $\frac{\mathrm{d}s}{\mathrm{d}t}$.

$\frac{\mathrm{d}s}{\mathrm{d}t} = 25\ (\mathrm{km\,h^{-1}})$ ***A1***

hence the rate is independent of time ***AG***

[5 marks]

Examiners report

Reasonably well done. Most successful candidates determined that $s = 25t \Rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} = 25$ from $x = 15t$ and $y = 20t$. A number of candidates did not use calculus while a few candidates correctly used implicit differentiation.

A bicycle inner tube can be considered as a joined up cylinder of fixed length 200 cm and radius r cm. The radius r increases as the inner tube is pumped up. Air is being pumped into the inner tube so that the volume of air in the tube increases at a constant rate of $30\ \mathrm{cm^3 s^{-1}}$. Find the rate at which the radius of the inner tube is increasing when $r = 2\ \mathrm{cm}$.

Markscheme

$V = 200\pi r^2$ ***(A1)***

Note: Allow $V = \pi h r^2$ if value of h is substituted later in the question.

EITHER

$\frac{\mathrm{d}V}{\mathrm{d}t} = 200\pi 2r \frac{\mathrm{d}r}{\mathrm{d}t}$ ***M1A1***

Note: Award ***M1*** for an attempt at implicit differentiation.

at $r = 2$ we have $30 = 200\pi 4 \frac{\mathrm{d}r}{\mathrm{d}t}$ ***M1***

OR

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}r}}$ ***M1***

$\frac{\mathrm{d}V}{\mathrm{d}r} = 400\pi r$ ***M1***

$r = 2$ we have $\frac{\mathrm{d}V}{\mathrm{d}r} = 800\pi$ ***A1***

THEN

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{30}{800\pi} \quad \left(= \frac{3}{80\pi} = 0.0119\right) \ (\mathrm{cm\,s^{-1}})$ ***A1***

[5 marks]

Examiners report

This question was well understood and a large percentage appreciated the need for implicit differentiation although some candidates did not recognise the need to treat h as a constant till late in the question. A number of candidates found the answer $\frac{3\pi}{80}$ instead of $\frac{3}{80\pi}$ due to a basic incorrect use of the GDC.

The region R is enclosed by the graph of $y = e^{-x^2}$, the x-axis and the lines $x = -1$ and $x = 1$.

Find the volume of the solid of revolution that is formed when R is rotated through 2π about the x-axis.

Markscheme

$\int_{-1}^{1} \pi \left(\mathrm{e}^{-x^2}\right)^2 \mathrm{d}x \quad \left(\int_{-1}^{1} \pi \mathrm{e}^{-2x^2} \mathrm{d}x \quad \text{or} \quad \int_{0}^{1} 2\pi \mathrm{e}^{-2x^2} \mathrm{d}x\right)$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for integral involving the function given; ***A1*** for correct limits; ***A1*** for π and $\left(\mathrm{e}^{-x^2}\right)^2$

$= 3.758249\ldots = 3.76$ ***A1***

[4 marks]

Examiners report

Most candidates answered this question correctly. Those candidates who attempted to manipulate the function or attempt an integration wasted time and obtained 3/4 marks. The most common errors were an extra factor ‘2’ and a fourth power when attempting to square the function. Many candidates wrote down the correct expression but not all were able to use their calculator correctly.

Consider the curve defined by the equation $4x^2 + y^2 = 7$.

a. Find the equation of the normal to the curve at the point $\left(1,\ \sqrt{3}\right)$. [6]

b. Find the volume of the solid formed when the region bounded by the curve, the x-axis for $x \geqslant 0$ and the y-axis for $y \geqslant 0$ is rotated through 2π about the x-axis. [3]

Markscheme

a. **METHOD 1**

$4x^2 + y^2 = 7$

$8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(M1)(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{y}$

Note: Award ***M1A1*** for finding $\frac{\mathrm{d}y}{\mathrm{d}x} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{y}{4x}$ ***(M1)***

hence gradient of normal at $\left(1,\ \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4}$ $(= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x-1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)$ $(y = 0.433x + 1.30)$

METHOD 2

$4x^2 + y^2 = 7$

$y = \sqrt{7 - 4x^2}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x}{\sqrt{7-4x^2}}$ ***(A1)***

Note: Award ***M1A1*** for finding $\frac{\mathrm{d}y}{\mathrm{d}x} = -2.309\ldots$ using any alternative method.

hence gradient of normal $= \frac{\sqrt{7-4x^2}}{4x}$ ***(M1)***

hence gradient of normal at $\left(1,\ \sqrt{3}\right)$ is $\frac{\sqrt{3}}{4}$ $(= 0.433)$ ***(A1)***

hence equation of normal is $y - \sqrt{3} = \frac{\sqrt{3}}{4}(x-1)$ ***(M1)A1***

$\left(y = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}}{4}\right)$ $(y = 0.433x + 1.30)$

[6 marks]

b. Use of $V = \pi \int\limits_0^{\frac{\sqrt{7}}{2}} y^2 \mathrm{d}x$

$V = \pi \int\limits_0^{\frac{\sqrt{7}}{2}} \left(7 - 4x^2\right) \mathrm{d}x$ ***(M1)(A1)***

Note: Condone absence of limits or incorrect limits for ***M*** mark.

Do not condone absence of or multiples of π.

$= 19.4$ $\left(= \frac{7\sqrt{7}\pi}{3}\right)$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

A. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$, [8]

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

B. (a) Using integration by parts, show that $\int \mathrm{e}^{2x} \sin x \mathrm{d}x = \frac{1}{5}\mathrm{e}^{2x}(2\sin x - \cos x) + C$. [17]

(b) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 - y^2}\mathrm{e}^{2x} \sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form $y = f(x)$.

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.

Calculate the volume of this solid.

Markscheme

A. prove that $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$

for $n = 1$

$\text{LHS} = 1, \ \text{RHS} = 4 - \frac{1+2}{2^0} = 4 - 3 = 1$

so true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

now for $n = k + 1$

LHS: $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ (or equivalent) ***A1***

$=4-\frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4-\frac{k+3}{2^k}$) ***A1***

Therefore if it is true for $n = k$ it is true for $n = k + 1$. It has been shown to be true for $n = 1$ so it is true for all n $(\in \mathbb{Z}^+)$. ***R1***

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

B. (a)

METHOD 1

$\int e^{2x}\sin x\,dx = -\cos x e^{2x} + \int 2e^{2x}\cos x\,dx$ ***M1A1A1***

$= -\cos x e^{2x} + 2e^{2x}\sin x - \int 4e^{2x}\sin x\,dx$ ***A1A1***

$5\int e^{2x}\sin x\,dx = -\cos x e^{2x} + 2e^{2x}\sin x$ ***M1***

$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$ ***AG***

METHOD 2

$\int \sin x e^{2x}dx = \frac{\sin x e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2}dx$ ***M1A1A1***

$= \frac{\sin x e^{2x}}{2} - \cos x \frac{e^{2x}}{4} - \int \sin x \frac{e^{2x}}{4}dx$ ***A1A1***

$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{e^{2x}\sin x}{2} - \frac{\cos x e^{2x}}{4}$ ***M1***

$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$ ***AG***

[6 marks]

(b)

$\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x}\sin x\,dx$ ***M1A1***

$\arcsin y = \frac{1}{5}e^{2x}(2\sin x - \cos x)(+C)$ ***A1***

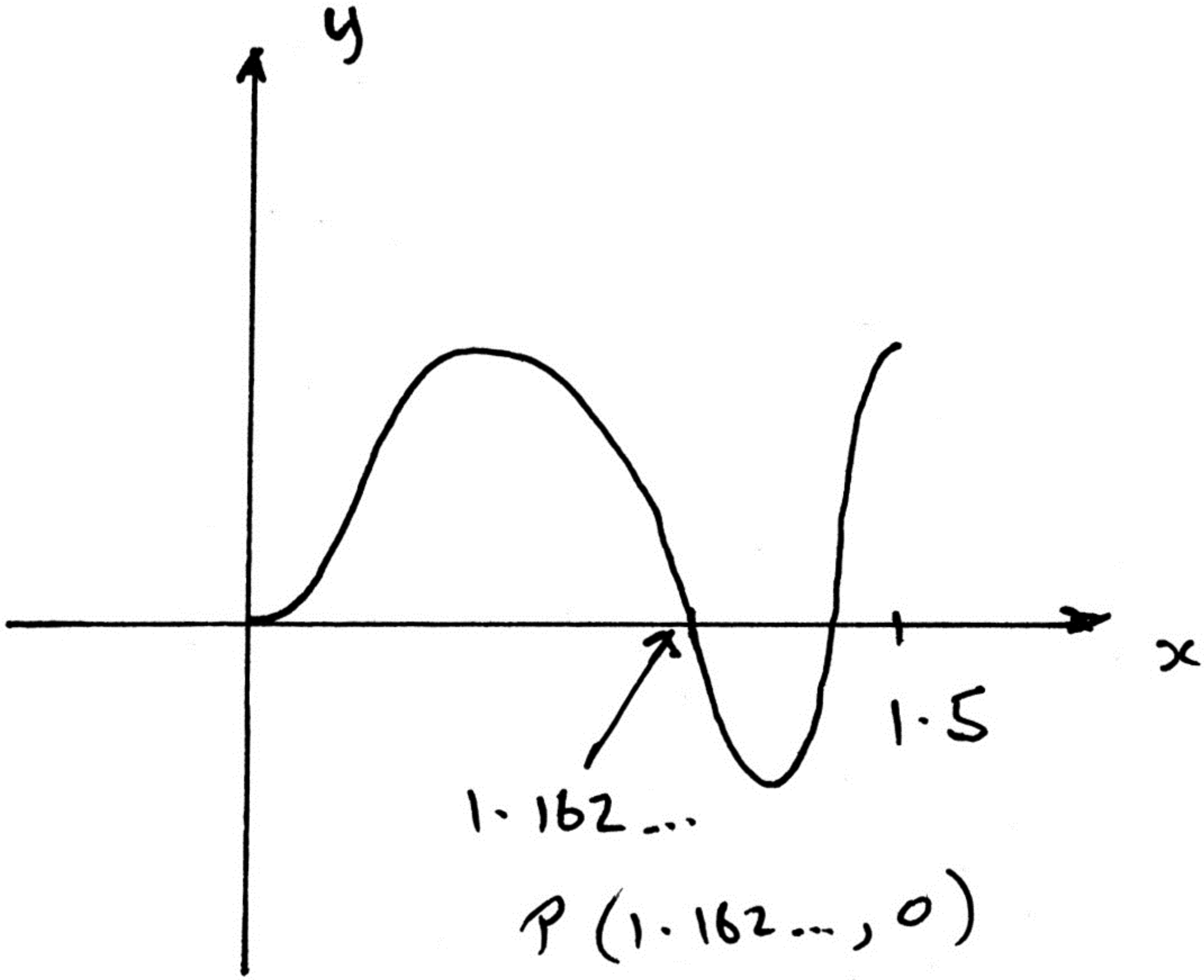
when $x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$ ***M1***

$y = \sin\left(\frac{1}{5}e^{2x}(2\sin x - \cos x) + \frac{1}{5}\right)$ ***A1***

[5 marks]

(c)

(i)



A1

P is (1.16, 0) *A1*

Note: Award *A1* for 1.16 seen anywhere, *A1* for complete sketch.

Note: Allow FT on their answer from (b)

(ii) $V = \int_0^{1.162...} \pi y^2 \mathrm{d}x$ ***M1A1***

$= 1.05$ ***A2***

Note: Allow FT on their answers from (b) and (c)(i).

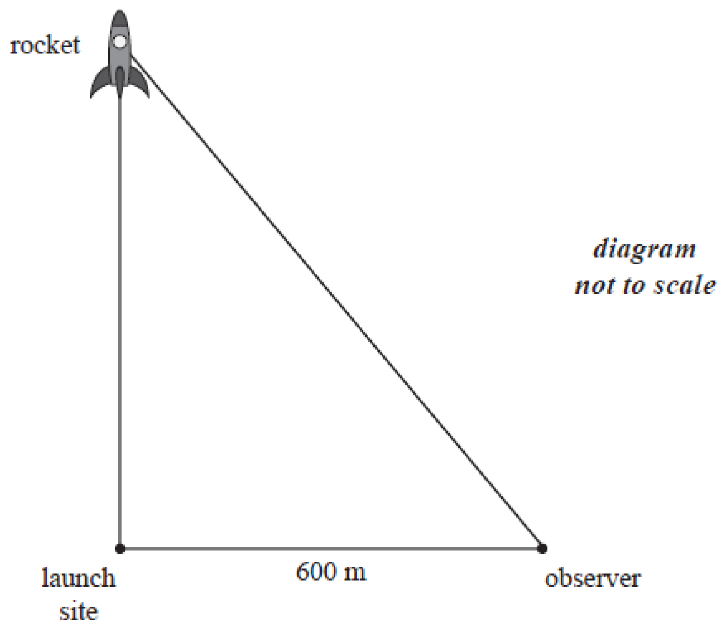
[6 marks]

Examiners report

A. Part A: Given that this question is at the easier end of the ‘proof by induction’ spectrum, it was disappointing that so many candidates failed to score full marks. The $n = 1$ case was generally well done. The whole point of the method is that it involves logic, so ‘let n = k’ or ‘put n = k’, instead of ‘assume ... to be true for n = k’, gains no marks. The algebraic steps need to be more convincing than some candidates were able to show. It is astonishing that the R1 mark for the final statement was so often not awarded.

B. Part B: Part (a) was often well done, although some faltered after the first integration. Part (b) was also generally well done, although there were some errors with the constant of integration. In (c) the graph was often attempted, but errors in (b) usually led to manifestly incorrect plots. Many attempted the volume of integration and some obtained the correct value.

A rocket is rising vertically at a speed of 300 ms^{-1} when it is 800 m directly above the launch site. Calculate the rate of change of the distance between the rocket and an observer, who is 600 m from the launch site and on the same horizontal level as the launch site.



Markscheme

let x = distance from observer to rocket

let h = the height of the rocket above the ground

METHOD 1

$\frac{\mathrm{d}h}{\mathrm{d}t} = 300$ when $h = 800$ ***A1***

$x = \sqrt{h^2 + 360\,000} = (h^2 + 360\,000)^{\frac{1}{2}}$ ***M1***

$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{h}{\sqrt{h^2+360\,000}}$ ***A1***

when h = 800

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1***

$= \frac{300h}{\sqrt{h^2+360\,000}}$ ***A1***

$= 240\ (\mathrm{ms}^{-1})$ ***A1***

[6 marks]

METHOD 2

$h^2 + 600^2 = x^2$ ***M1***

$2h = 2x\frac{\mathrm{d}x}{\mathrm{d}h}$ ***A1***

$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{h}{x}$

$= \frac{800}{1000}\left(= \frac{4}{5}\right)$ ***A1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = 300$ ***A1***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1***

$= \frac{4}{5} \times 300$

$= 240\ (\mathrm{ms}^{-1})$ ***A1***

[6 marks]

METHOD 3

$x^2 = 600^2 + h^2$ ***M1***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} = 2h\frac{\mathrm{d}h}{\mathrm{d}t}$ ***A1A1***

when $h = 800$, $x = 1000$

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{800}{1000} \times \frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1A1***

$= 240\ (\mathrm{ms}^{-1})$ ***A1***

[6 marks]

METHOD 4

Distance between the observer and the rocket $= (600^2 + 800^2)^{\frac{1}{2}} = 1000$ ***M1A1***

Component of the velocity in the line of sight $= \sin\theta \times 300$

(where $\theta =$ angle of elevation) ***M1A1***

$\sin\theta = \frac{800}{1000}$ ***A1***

component $= 240\ (\mathrm{ms}^{-1})$ ***A1***

[6 marks]

Examiners report

Questions of this type are often open to various approaches, but most full solutions require the application of ‘related rates of change’. Although most candidates realised this, their success rate was low. This was particularly apparent in approaches involving trigonometric functions. Some candidates assumed constant speed – this gained some small credit. Candidates should be encouraged to state what their symbols stand for.

A particle moves such that its velocity $v\,\mathrm{ms}^{-1}$ is related to its displacement $s\,\mathrm{m}$, by the equation $v(s) = \arctan(\sin s),\ 0 \leqslant s \leqslant 1$. The particle’s acceleration is $a\,\mathrm{ms}^{-2}$.

a. Find the particle's acceleration in terms of s. [4]

b. Using an appropriate sketch graph, find the particle's displacement when its acceleration is $0.25\ \mathrm{ms}^{-2}$. [2]

Markscheme

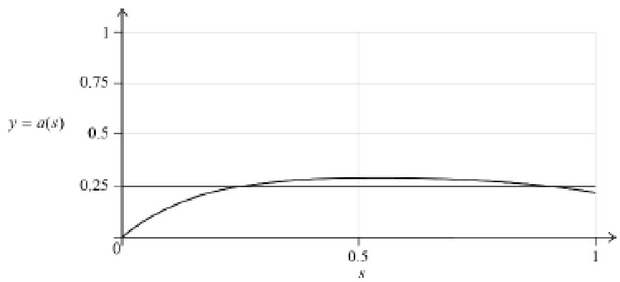
a. $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\cos s}{\sin^2 s+1}$ ***M1A1***

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$ ***(M1)***

$a = \frac{\arctan(\sin s)\cos s}{\sin^2 s+1}$ ***A1***

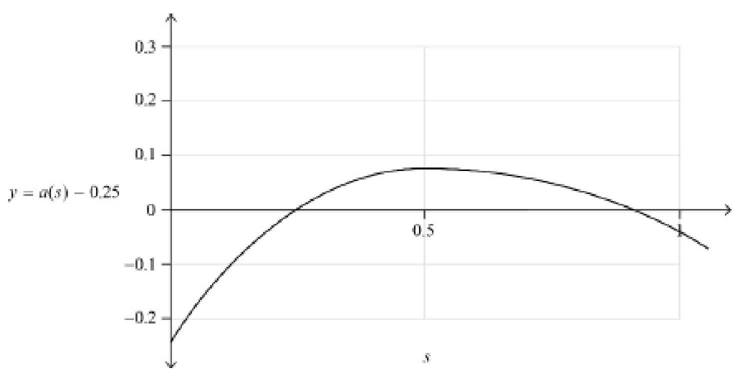
[4 marks]

b. **EITHER**



(M1)

OR



(M1)

THEN

$s = 0.296,\ 0.918\ (\mathrm{m})$ ***A1***

[2 marks]

Examiners report

a. In part (a), a large number of candidates thought that $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}s}$ rather than $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t} \times \frac{\mathrm{d}v}{\mathrm{d}s} = v\frac{\mathrm{d}v}{\mathrm{d}s}$.

b. In part (b), quite a few of these candidates then went on to find a value of s that was outside the domain $0 \leqslant s \leqslant 1$.

The function f has inverse f^{-1} and derivative $f'(x)$ for all $x \in \mathbb{R}$. For all functions with these properties you are given the result that for $a \in \mathbb{R}$ with $b = f(a)$ and $f'(a) \neq 0$

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

a. Verify that this is true for $f(x) = x^3 + 1$ at $x = 2$. [6]

b. Given that $g(x) = xe^{x^2}$, show that $g'(x) > 0$ for all values of x. [3]

c. Using the result given at the start of the question, find the value of the gradient function of $y = g^{-1}(x)$ at $x = 2$. [4]

d. (i) With f and g as defined in parts (a) and (b), solve $g \circ f(x) = 2$. [6]

(ii) Let $h(x) = (g \circ f)^{-1}(x)$. Find $h'(2)$.

Markscheme

a. $f(2) = 9$ ***(A1)***

$f^{-1}(x) = (x-1)^{\frac{1}{3}}$ ***A1***

$(f^{-1})'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$ ***(M1)***

$(f^{-1})'(9) = \frac{1}{12}$ ***A1***

$f'(x) = 3x^2$ ***(M1)***

$\frac{1}{f'(2)} = \frac{1}{3 \times 4} = \frac{1}{12}$ ***A1***

Note: The last ***M1*** and ***A1*** are independent of previous marks.

[6 marks]

b. $g'(x) = e^{x^2} + 2x^2e^{x^2}$ ***M1A1***

$g'(x) > 0$ as each part is positive ***R1***

[3 marks]

c. to find the x-coordinate on $y = g(x)$ solve

$2 = xe^{x^2}$ ***(M1)***

$x = 0.89605022078\ldots$ ***(A1)***

gradient $= (g^{-1})'(2) = \frac{1}{g'(0.896\ldots)}$ ***(M1)***

$= \frac{1}{e^{(0.896\ldots)^2}\left(1+2\times(0.896\ldots)^2\right)} = 0.172$ to 3sf ***A1***

(using the $\frac{dy}{dx}$ function on gdc $g'(0.896\ldots) = 5.7716028\ldots$

$\frac{1}{g'(0.896\ldots)} = 0.173$

[4 marks]

d. (i) $(x^3+1)e^{(x^3+1)^2} = 2$ ***A1***

$x = -0.470191\ldots$ ***A1***

(ii) **METHOD 1**

$(g \circ f)'(x) = 3x^2e^{(x^3+1)^2}\left(2(x^3+1)^2+1\right)$ ***(M1)(A1)***

$(g \circ f)'(-0.470191\ldots) = 3.85755\ldots$ ***(A1)***

$h'(2) = \frac{1}{3.85755\ldots} = 0.259\ (232\ldots)$ ***A1***

Note: The solution can be found without the student obtaining the explicit form of the composite function.

METHOD 2

$h(x) = (f^{-1} \circ g^{-1})(x)$ ***A1***

$h'(x) = (f^{-1})'\left(g^{-1}(x)\right) \times (g^{-1})'(x)$ ***M1***

$= \frac{1}{3}\left(g^{-1}(x) - 1\right)^{-\frac{2}{3}} \times (g^{-1})'(x)$ ***M1***

$h'(2) = \frac{1}{3}\left(g^{-1}(2) - 1\right)^{-\frac{2}{3}} \times (g^{-1})'(2)$

$= \frac{1}{3}(0.89605\ldots - 1)^{-\frac{2}{3}} \times 0.171933\ldots$

$= 0.259\ (232\ldots)$ ***A1*** ***N4***

[6 marks]

Examiners report

a. There were many good attempts at parts (a) and (b), although in (b) many were unable to give a thorough justification.

b. There were many good attempts at parts (a) and (b), although in (b) many were unable to give a thorough justification.

c. Few good solutions to parts (c) and (d)(ii) were seen although many were able to answer (d)(i) correctly.

d. Few good solutions to parts (c) and (d)(ii) were seen although many were able to answer (d)(i) correctly.

A family of cubic functions is defined as $f_k(x) = k^2x^3 - kx^2 + x,\ k \in \mathbb{Z}^+$.

(a) Express in terms of k

(i) $f'_k(x)$ and $f''_k(x)$;

(ii) the coordinates of the points of inflexion P_k on the graphs of f_k .

(b) Show that all P_k lie on a straight line and state its equation.

(c) Show that for all values of k, the tangents to the graphs of f_k at P_k are parallel, and find the equation of the tangent lines.

Markscheme

(a) (i) $f'_k(x) = 3k^2x^2 - 2kx + 1$ ***A1***

$f''_k(x) = 6k^2x - 2k$ ***A1***

(ii) Setting $f''(x) = 0$ ***M1***

$\Rightarrow 6k^2x - 2k = 0 \Rightarrow x = \frac{1}{3k}$ ***A1***

$f\left(\frac{1}{3k}\right) = k^2\left(\frac{1}{3k}\right)^3 - k\left(\frac{1}{3k}\right)^2 + \left(\frac{1}{3k}\right)$ ***M1***

$= \frac{7}{27k}$ ***A1***

Hence, P_k is $\left(\frac{1}{3k}, \frac{7}{27k}\right)$

[6 marks]

(b) Equation of the straight line is $y = \frac{7}{9}x$ ***A1***

As this equation is independent of k, all P_k lie on this straight line ***R1***

[2 marks]

(c) Gradient of tangent at P_k :

$f'(P_k) = f'\left(\frac{1}{3k}\right) = 3k^2\left(\frac{1}{3k}\right)^2 - 2k\left(\frac{1}{3k}\right) + 1 = \frac{2}{3}$ ***M1A1***

As the gradient is independent of *k*, the tangents are parallel. ***R1***

$\frac{7}{27k} = \frac{2}{3} \times \frac{1}{3k} + c \Rightarrow c = \frac{1}{27k}$ ***(A1)***

The equation is $y = \frac{2}{3}x + \frac{1}{27k}$ ***A1***

[5 marks]

Total [13 marks]

Examiners report

Many candidates scored the full 6 marks for part (a). The main mistake evidenced was to treat *k* as a variable, and hence use the product rule to differentiate. Of the many candidates who attempted parts (b) and (c), few scored the R1 marks in either part, but did manage to get the equations of the straight lines.

A triangle is formed by the three lines $y = 10 - 2x,\ y = mx$ and $y = -\frac{1}{m}x$, where $m > \frac{1}{2}$.

Find the value of *m* for which the area of the triangle is a minimum.

Markscheme

attempt to find intersections ***M1***

intersections are $\left(\frac{10}{m+2}, \frac{10m}{m+2}\right)$ and $\left(\frac{10m}{2m-1}, -\frac{10}{2m-1}\right)$ ***A1A1***

area of triangle $= \frac{1}{2} \times \frac{\sqrt{100+100m^2}}{(m+2)} \times \frac{\sqrt{100+100m^2}}{(2m-1)}$ ***M1A1A1***

$= \frac{50(1+m^2)}{(m+2)(2m-1)}$

minimum when $m = 3$ ***(M1)A1***

[8 marks]

Examiners report

Most candidates had difficulties with this question and did not go beyond the determination of the intersection points of the lines; in a few cases candidates set up the expression of the area, in some cases using unsimplified expressions of the coordinates.

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s^{-1}}$, at time *t* seconds, is given by

$$v(t) = \begin{cases} 5 - (t-2)^2, & 0 \le t \le 4 \\ 3 - \frac{t}{2}, & t > 4 \end{cases}.$$

a. Find the value of t when the particle is instantaneously at rest. [2]

b. The particle returns to its initial position at $t = T$. [5]

Find the value of *T*.

Markscheme

a. $3 - \frac{t}{2} = 0 \Rightarrow t = 6\ (\mathrm{s})$ ***(M1)A1***

Note: Award ***A0*** if either $t = -0.236$ or $t = 4.24$ or both are stated with $t = 6$.

[2 marks]

b. let d be the distance travelled before coming to rest

$d = \int_0^4 5 - (t-2)^2 \mathrm{d}t + \int_4^6 3 - \frac{t}{2} \mathrm{d}t$ ***(M1)(A1)***

Note: Award ***M1*** for two correct integrals even if the integration limits are incorrect. The second integral can be specified as the area of a triangle.

$d = \frac{47}{3}\quad (= 15.7)\ (\mathrm{m})$ ***(A1)***

attempting to solve $\int_6^T \left(\frac{t}{2} - 3\right) \mathrm{d}t = \frac{47}{3}$ (or equivalent) for T ***M1***

$T = 13.9\ (\mathrm{s})$ ***A1***

[5 marks]

Total [7 marks]

Examiners report

a. Part (a) was not done as well as expected. A large number of candidates attempted to solve $5 - (t-2)^2 = 0$ for t. Some candidates attempted to find when the particle's acceleration was zero.

b. Most candidates had difficulty with part (b) with a variety of errors committed. A significant proportion of candidates did not understand what was required. Many candidates worked with indefinite integrals rather than with definite integrals. Only a small percentage of candidates started by correctly finding the distance travelled by the particle before coming to rest. The occasional candidate made adroit use of a GDC and found the correct value of t by finding where the graph of $\int_0^4 5 - (t-2)^2 \mathrm{d}t + \int_4^x 3 - \frac{t}{2} \mathrm{d}t$ crossed the horizontal axis.

Let $f(x) = \frac{a + b\mathrm{e}^x}{a\mathrm{e}^x + b}$, where $0 < b < a$.

(a) Show that $f'(x) = \frac{(b^2 - a^2)\mathrm{e}^x}{(a\mathrm{e}^x + b)^2}$.

(b) **Hence** justify that the graph of *f* has no local maxima or minima.

(c) Given that the graph of *f* has a point of inflexion, find its coordinates.

(d) Show that the graph of *f* has exactly two asymptotes.

(e) Let *a* = 4 and *b* =1. Consider the region *R* enclosed by the graph of $y = f(x)$, the *y*-axis and the line with equation $y = \frac{1}{2}$.

Find the volume V of the solid obtained when R is rotated through 2π about the x-axis.

Markscheme

(a) $f'(x) = \frac{be^x(ae^x+b)-ae^x(a+be^x)}{(ae^x+b)^2}$ ***M1A1***

$= \frac{abe^{2x}+b^2e^x-a^2e^x-abe^{2x}}{(ae^x+b)^2}$ ***A1***

$= \frac{(b^2-a^2)e^x}{(ae^x+b)^2}$ ***AG***

[3 marks]

(b) ***EITHER***

$f'(x) = 0 \Rightarrow (b^2-a^2)e^x = 0 \Rightarrow b = \pm a$ or $e^x = 0$ ***A1***

which is impossible as $0 < b < a$ and $e^x > 0$ for all $x \in \mathbb{R}$ ***R1***

OR

$f'(x) < 0$ for all $x \in \mathbb{R}$ since $0 < b < a$ and $e^x > 0$ for all $x \in \mathbb{R}$ ***A1R1***

OR

$f'(x)$ cannot be equal to zero because e^x is never equal to zero ***A1R1***

[2 marks]

(c) **EITHER**

$f''(x) = \frac{(b^2-a^2)e^x(ae^x+b)^2-2ae^x(ae^x+b)(b^2-a^2)e^x}{(ae^x+b)^4}$ ***M1A1A1***

Note: Award ***A1*** for each term in the numerator.

$= \frac{(b^2-a^2)e^x(ae^x+b-2ae^x)}{(ae^x+b)^3}$

$= \frac{(b^2-a^2)(b-ae^x)e^x}{(ae^x+b)^3}$

OR

$f'(x) = (b^2-a^2)e^x(ae^x+b)^{-2}$

$f''(x) = (b^2-a^2)e^x(ae^x+b)^{-2} + (b^2-a^2)e^x(-2ae^x)(ae^x+b)^{-3}$ ***M1A1A1***

Note: Award ***A1*** for each term.

$= (b^2-a^2)e^x(ae^x+b)^{-3}((ae^x+b)-2ae^x)$

$= (b^2-a^2)e^x(ae^x+b)^{-3}(b-ae^x)$

THEN

$f''(x) = 0 \Rightarrow b - ae^x = 0 \Rightarrow x = \ln\frac{b}{a}$ ***M1A1***

$f\left(\ln\frac{b}{a}\right) = \frac{a^2+b^2}{2ab}$ ***A1***

coordinates are $\left(\ln\frac{b}{a}, \frac{a^2+b^2}{2ab}\right)$

[6 marks]

(d) $\lim\limits_{x-\infty} f(x) = \frac{a}{b} \Rightarrow y = \frac{a}{b}$ horizontal asymptote ***A1***

$\lim\limits_{x \to +\infty} f(x) = \frac{b}{a} \Rightarrow y = \frac{b}{a}$ horizontal asymptote ***A1***

$0 < b < a \Rightarrow a\mathrm{e}^x + b > 0$ for all $x \in \mathbb{R}$ (accept $a\mathrm{e}^x + b \neq 0$)

so no vertical asymptotes ***R1***

Note: Statement on vertical asymptote must be seen for ***R1***.

[3 marks]

(e) $y = \frac{4+\mathrm{e}^x}{4\mathrm{e}^x+1}$

$y = \frac{1}{2} \Leftrightarrow x = \ln\frac{7}{2}$ (or 1.25 to 3 sf) ***(M1)(A1)***

$V = \pi \int_0^{\ln\frac{7}{2}} \left(\left(\frac{4+\mathrm{e}^x}{4\mathrm{e}^x+1} \right)^2 - \frac{1}{4} \right) \mathrm{d}x$ ***(M1)A1***

$= 1.09$ (3 sf) ***A1*** ***N4***

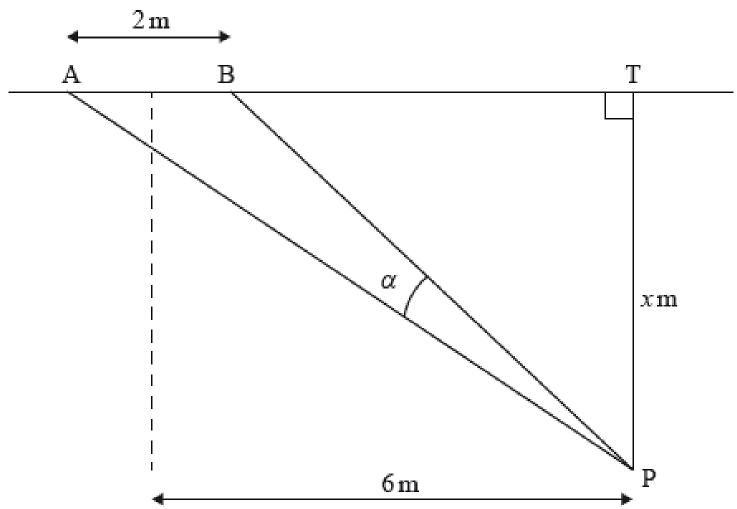
[5 marks]

Total [19 marks]

Examiners report

This question was well attempted by many candidates. In some cases, candidates who skipped other questions still answered, with some success, parts of this question. Part (a) was in general well done but in (b) candidates found difficulty in justifying that f'(x) was non-zero. Performance in part (c) was mixed: it was pleasing to see good levels of algebraic ability of good candidates who successfully answered this question; weaker candidates found the simplification required difficult. There were very few good answers to part (d) which showed the weaknesses of most candidates in dealing with the concept of asymptotes. In part (e) there were a large number of good attempts, with many candidates evaluating correctly the limits of the integral and a smaller number scoring full marks in this part.

Points A , B and T lie on a line on an indoor soccer field. The goal, [AB] , is 2 metres wide. A player situated at point P kicks a ball at the goal. [PT] is perpendicular to (AB) and is 6 metres from a parallel line through the centre of [AB] . Let PT be x metros and let $\alpha = \mathrm{A\hat{P}B}$ measured in degrees. Assume that the ball travels along the floor.



The maximum for $\tan\alpha$ gives the maximum for α.

a. Find the value of α when $x = 10$. [4]

b. Show that $\tan\alpha = \frac{2x}{x^2+35}$. [4]

c. (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha)$. [11]

(ii) Hence or otherwise find the value of α such that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0$.

(iii) Find $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ and hence show that the value of α never exceeds 10°.

d. Find the set of values of x for which $\alpha \geqslant 7^\circ$. [3]

Markscheme

a. **EITHER**

$\alpha = \arctan\frac{7}{10} - \arctan\frac{5}{10}\ (= 34.992\ldots^\circ - 26.5651\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{A\hat{P}T} - \mathrm{B\hat{P}T}$, ***(A1)*** for a correct $\mathrm{A\hat{P}T}$ and ***(A1)*** for a correct $\mathrm{B\hat{P}T}$.

OR

$\alpha = \arctan\ 2 - \arctan\frac{10}{7}\ (= 63.434\ldots^\circ - 55.008\ldots^\circ)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for $\alpha = \mathrm{P\hat{B}T} - \mathrm{P\hat{A}T}$, ***(A1)*** for a correct $\mathrm{P\hat{B}T}$ and ***(A1)*** for a correct $\mathrm{P\hat{A}T}$.

OR

$\alpha = \arccos\left(\frac{125+149-4}{2\times\sqrt{125}\times\sqrt{149}}\right)$ ***(M1)(A1)(A1)***

Note: Award ***(M1)*** for use of cosine rule, ***(A1)*** for a correct numerator and ***(A1)*** for a correct denominator.

THEN

$= 8.43^\circ$ ***A1***

[4 marks]

b. **EITHER**

$\tan\alpha = \frac{\frac{7}{x}-\frac{5}{x}}{1+\left(\frac{7}{x}\right)\left(\frac{5}{x}\right)}$ ***M1A1A1***

Note: Award ***M1*** for use of $\tan(A-B)$, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$= \frac{\frac{2}{x}}{1+\frac{35}{x^2}}$ ***M1***

OR

$\tan\alpha = \frac{\frac{x}{5}-\frac{x}{7}}{1+\left(\frac{x}{5}\right)\left(\frac{x}{7}\right)}$ ***M1A1A1***

Note: Award ***M1*** for use of xxx, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$= \frac{\frac{2x}{35}}{1+\frac{x^2}{35}}$ ***M1***

OR

$\cos\alpha = \frac{x^2+35}{\sqrt{(x^2+25)(x^2+49)}}$ ***M1A1***

Note: Award ***M1*** for either use of the cosine rule or use of $\cos(A-B)$.

$\sin\alpha \frac{2x}{\sqrt{(x^2+25)(x^2+49)}}$ ***A1***

$\tan\alpha = \frac{\frac{2x}{\sqrt{(x^2+25)(x^2+49)}}}{\frac{x^2+35}{\sqrt{(x^2+25)(x^2+49)}}}$ ***M1***

THEN

$\tan\alpha = \frac{2x}{x^2+35}$ ***AG***

[4 marks]

c. (i) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = \frac{2(x^2+35)-(2x)(2x)}{(x^2+35)^2} \left(= \frac{70-2x^2}{(x^2+35)^2}\right)$ ***M1A1A1***

Note: Award ***M1*** for attempting product or quotient rule differentiation, ***A1*** for a correct numerator and ***A1*** for a correct denominator.

(ii) **METHOD 1**

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = 0 \Rightarrow 70-2x^2 = 0$ ***(M1)***

$x = \sqrt{35}\ (\mathrm{m})\ (= 5.9161\ldots\ (\mathrm{m}))$ ***A1***

$\tan\alpha = \frac{1}{\sqrt{35}}\ (= 0.16903\ldots)$ ***(A1)***

OR

attempting to locate the stationary point on the graph of

$\tan\alpha = \frac{2x}{x^2+35}$ ***(M1)***

$x = 5.9161\ldots\ (\mathrm{m})\ \left(= \sqrt{35}\ (\mathrm{m})\right)$ ***A1***

$\tan\alpha = 0.16903\ldots \left(= \frac{1}{\sqrt{35}}\right)$ ***(A1)***

THEN

$\alpha = 9.59^\circ$ ***A1***

METHOD 2

EITHER

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right) \Rightarrow \frac{\mathrm{d}\alpha}{\mathrm{d}x} = \frac{70-2x^2}{(x^2+35)^2+4x^2}$ ***M1***

$\frac{\mathrm{d}\alpha}{\mathrm{d}x} = 0 \Rightarrow x = \sqrt{35}\ (\mathrm{m})\ (= 5.9161\ (\mathrm{m}))$ ***A1***

OR

attempting to locate the stationary point on the graph of

$\alpha = \arctan\left(\frac{2x}{x^2+35}\right)$ ***(M1)***

$x = 5.9161\ldots\ (\mathrm{m})\ \left(= \sqrt{35}\,(\mathrm{m})\right)$ ***A1***

THEN

$\alpha = 0.1674\ldots\ \left(= \arctan\frac{1}{\sqrt{35}}\right)$ ***(A1)***

$= 9.59^\circ$ ***A1***

(iii) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha) = \frac{(x^2+25)^2(-4x)-(2)(2x)(x^2+35)(70-2x^2)}{(x^2+35)^4}\ \left(= \frac{4x(x^2-105)}{(x^2+35)^3}\right)$ ***M1A1***

substituting $x = \sqrt{35}\ (= 5.9161\ldots)$ into $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ ***M1***

$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha) < 0\ (= -0.004829\ldots)$ and so $\alpha = 9.59^\circ$ is the maximum value of α ***R1***

α never exceeds 10° ***AG***

[11 marks]

d. attempting to solve $\frac{2x}{x^2+35} \geqslant \tan 7^\circ$ ***(M1)***

Note: Award ***(M1)*** for attempting to solve $\frac{2x}{x^2+35} = \tan 7^\circ$.

$x = 2.55$ and $x = 13.7$ ***(A1)***

$2.55 \leqslant x \leqslant 13.7\ (\mathrm{m})$ ***A1***

[3 marks]

Examiners report

a. This question was generally accessible to a large majority of candidates. It was pleasing to see a number of different (and quite clever) trigonometric methods successfully employed to answer part (a) and part (b).

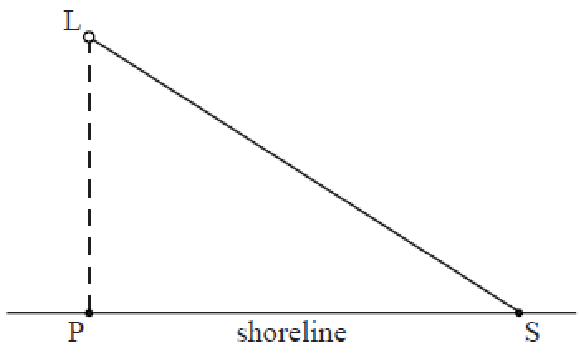
b. This question was generally accessible to a large majority of candidates. It was pleasing to see a number of different (and quite clever) trigonometric methods successfully employed to answer part (a) and part (b).

c. The early parts of part (c) were generally well done. In part (c) (i), a few candidates correctly found $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha)$ in unsimplified form but then committed an algebraic error when endeavouring to simplify further. A few candidates merely stated that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan\alpha) = \sec^2\alpha$.

Part (c) (ii) was reasonably well done with a large number of candidates understanding what was required to find the correct value of α in degrees. In part (c)(iii), a reasonable number of candidates were able to successfully find $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ in unsimplified form. Some however attempted to solve $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha) = 0$ for χ rather than examine the value of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tan\alpha)$ at $x = \sqrt{35}$.

d. Part (d), which required use of a GCD to determine an inequality, was surprisingly often omitted by candidates. Of the candidates who attempted this part, a number stated that $x \geqslant 2.55$. Quite a sizeable proportion of candidates who obtained the correct inequality did not express their answer to 3 significant figures.

A lighthouse L is located offshore, 500 metres from the nearest point P on a long straight shoreline. The narrow beam of light from the lighthouse rotates at a constant rate of 8π radians per minute, producing an illuminated spot S that moves along the shoreline. You may assume that the height of the lighthouse can be ignored and that the beam of light lies in the horizontal plane defined by sea level.



When S is 2000 metres from P,

(a) show that the speed of S, correct to three significant figures, is 214 000 metres per minute;

(b) find the acceleration of S.

Markscheme

(a) the distance of the spot from P is $x = 500\tan\theta$ ***A1***

the speed of the spot is

$\frac{\mathrm{d}x}{\mathrm{d}t} = 500\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\ (= 4000\pi\sec^2\theta)$ ***M1A1***

when $x = 2000,\ \sec^2\theta = 17\ (\theta = 1.32581\ldots)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} = 8\pi\right)$

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 500 \times 17 \times 8\pi$ ***M1A1***

speed is 214 000 (metres per minute) ***AG***

Note: If their displayed answer does not round to 214 000, they lose the final ***A1***.

(b) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 8000\pi\sec^2\theta\tan\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$ **or** $500 \times 2\sec^2\theta\tan\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$ ***M1A1***

$$\left(\text{since } \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 0\right)$$

$= 43\,000\,000\ (= 4.30 \times 10^7)$ (metres per minute^2) ***A1***

[8 marks]

Examiners report

This was a wordy question with a clear diagram, requiring candidates to state variables and do some calculus. Very few responded appropriately.

Using the substitution $x=2\sin\theta$, show that

$$\int\sqrt{4-x^2}\mathrm{d}x=Ax\sqrt{4-x^2}+B\arcsin\frac{x}{2}+\text{constant},$$

where A and B are constants whose values you are required to find.

Markscheme

$\int\sqrt{4-x^2}\mathrm{d}x$

$x=2\sin\theta$

$\mathrm{d}x=2\cos\theta\mathrm{d}\theta$ ***A1***

$=\int\sqrt{4-4\sin^2\theta}\times 2\cos\theta\mathrm{d}\theta$ ***M1A1***

$=\int 2\cos\theta\times 2\cos\theta\mathrm{d}\theta$

$=4\int\cos^2\theta\mathrm{d}\theta$

now $\int\cos^2\theta\mathrm{d}\theta$

$=\int\left(\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$ ***M1A1***

$=\left(\frac{\sin 2\theta}{4}+\frac{1}{2}\theta\right)$ ***A1***

so original integral

$=2\sin 2\theta+2\theta$

$=2\sin\theta\cos\theta+2\theta$

$=\left(2\times\frac{x}{2}\times\frac{\sqrt{4-x^2}}{2}\right)+2\arcsin\left(\frac{x}{2}\right)$

$=\frac{x\sqrt{4-x^2}}{2}+2\arcsin\left(\frac{x}{2}\right)+C$ ***A1A1***

Note: Do not penalise omission of C.

$\left(A=\frac{1}{2},\ B=2\right)$

[8 marks]

Examiners report

For many candidates this was an all or nothing question. Examiners were surprised at the number of candidates who were unable to change the variable in the integral using the given substitution. Another stumbling block, for some candidates, was a lack of care with the application of the trigonometric version of Pythagoras' Theorem to reduce the integrand to a multiple of $\cos^2\theta$. However, candidates who obtained the latter were generally successful in completing the question.

A particle moves in a straight line in a positive direction from a fixed point O.

The velocity v m s^{-1} , at time t seconds, where $t\geqslant 0$, satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{-v(1+v^2)}{50}.$$

The particle starts from O with an initial velocity of 10 m s^{-1} .

(a) (i) Express as a definite integral, the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1} .

(ii) **Hence** calculate the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1} .

(b) (i) Show that, when $v>0$, the motion of this particle can also be described by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{-(1+v^2)}{50}$ where *x* metres is the displacement from O.

(ii) Given that *v* =10 when *x* = 0 , solve the differential equation expressing *x* in terms of *v*.

(iii) **Hence** show that $v=\frac{10-\tan\frac{x}{50}}{1+10\tan\frac{x}{50}}$.

Markscheme

(a) (i) **EITHER**

Attempting to separate the variables ***(M1)***

$\frac{\mathrm{d}v}{-v(1+v^2)}=\frac{\mathrm{d}t}{50}$ ***(A1)***

OR

Inverting to obtain $\frac{\mathrm{d}t}{\mathrm{d}v}$ ***(M1)***

$\frac{\mathrm{d}t}{\mathrm{d}v}=\frac{-50}{v(1+v^2)}$ ***(A1)***

THEN

$t=-50\int_{10}^{5}\frac{1}{v(1+v^2)}\mathrm{d}v\quad\left(=50\int_{5}^{10}\frac{1}{v(1+v^2)}\mathrm{d}v\right)$ ***A1 N3***

(ii) $t=0.732\ (\text{sec})\quad\left(=25\ln\frac{104}{101}(\text{sec})\right)$ ***A2 N2***

[5 marks]

(b) (i) $\frac{\mathrm{d}v}{\mathrm{d}t}=v\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

Must see division by *v* $(v>0)$ ***A1***

$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{-(1+v^2)}{50}$ ***AG N0***

(ii) Either attempting to separate variables or inverting to obtain $\frac{\mathrm{d}x}{\mathrm{d}v}$ ***(M1)***

$\frac{\mathrm{d}v}{1+v^2}=-\frac{1}{50}\int\mathrm{d}x$ (or equivalent) ***A1***

Attempting to integrate both sides ***M1***

$\arctan v=-\frac{x}{50}+C$ ***A1A1***

Note: Award ***A1*** for a correct LHS and ***A1*** for a correct RHS that must include *C*.

When $x=0$, $v=10$ and so $C=\arctan 10$ ***M1***

$x=50(\arctan 10-\arctan v)$ ***A1 N1***

(iii) Attempting to make $\arctan v$ the subject. ***M1***

$\arctan v=\arctan 10-\frac{x}{50}$ **A1**

$v=\tan\left(\arctan 10-\frac{x}{50}\right)$ ***M1A1***

Using $\tan(A-B)$ formula to obtain the desired form. ***M1***

$v=\frac{10-\tan\frac{x}{50}}{1+10\tan\frac{x}{50}}$ ***AG N0***

[14 marks]

Total [19 marks]

Examiners report

No comment.

Find the gradient of the curve $\mathrm{e}^{xy} + \ln(y^2) + \mathrm{e}^y = 1 + \mathrm{e}$ at the point (0, 1) .

Markscheme

$\mathrm{e}^{xy} + \ln(y^2) + \mathrm{e}^y = 1 + \mathrm{e}$

$\mathrm{e}^{xy}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{2}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, at (0, 1) ***A1A1A1A1A1***

$1\,(1+0) + 2\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$1 + 2\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2+\mathrm{e}}$ $(= -0.212)$ ***M1A1*** ***N2***

[7 marks]

Examiners report

Implicit differentiation is usually found to be difficult, but on this occasion there were many correct solutions. There were also a number of errors in the differentiation of e^{xy} , and although these often led to the correct final answer, marks could not be awarded.

A water trough which is 10 metres long has a uniform cross-section in the shape of a semicircle with radius 0.5 metres. It is partly filled with water as shown in the following diagram of the cross-section. The centre of the circle is O and the angle KOL is θ radians.

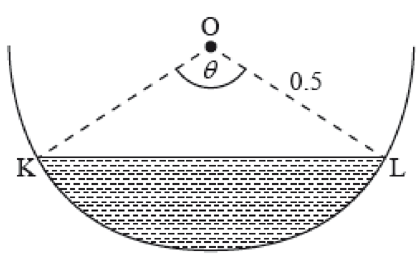


diagram not to scale

The volume of water is increasing at a constant rate of $0.0008\ \mathrm{m^3 s^{-1}}$.

a. Find an expression for the volume of water $V\ (\mathrm{m}^3)$ in the trough in terms of θ. [3]

b. Calculate $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ when $\theta = \frac{\pi}{3}$. [4]

Markscheme

a. area of segment $= \frac{1}{2} \times 0.5^2 \times (\theta - \sin\theta)$ ***M1A1***

$V = \text{area of segment} \times 10$

$V = \frac{5}{4}(\theta - \sin\theta)$ ***A1***

[3 marks]

b. **METHOD 1**

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{5}{4}(1 - \cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***M1A1***

$0.0008 = \frac{5}{4}\left(1 - \cos\frac{\pi}{3}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.00128\ (\text{rad}\ s^{-1})$ ***A1***

METHOD 2

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{5}{4}(1 - \cos\theta)$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{4 \times 0.0008}{5\left(1 - \cos\frac{\pi}{3}\right)}$ ***(M1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.00128\left(\frac{4}{3125}\right)(\text{rad}\ s^{-1})$ ***A1***

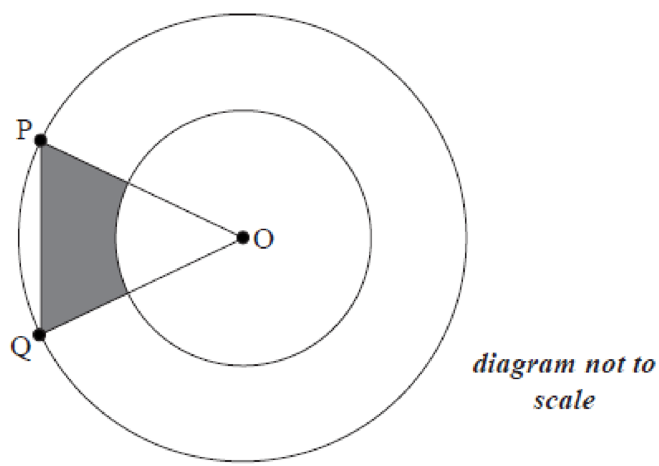
[4 marks]

Examiners report

a. [N/A]
b. [N/A]

The diagram below shows two concentric circles with centre O and radii 2 cm and 4 cm.

The points P and Q lie on the larger circle and $\mathrm{P\hat{O}Q} = x$, where $0 < x < \frac{\pi}{2}$.



(a) Show that the area of the shaded region is $8\sin x - 2x$.

(b) Find the maximum area of the shaded region.

Markscheme

(a) shaded area area of triangle area of sector, *i.e.* ***(M1)***

$\left(\frac{1}{2} \times 4^2 \sin x\right) - \left(\frac{1}{2}2^2 x\right) = 8 \sin x - 2x$ ***A1A1AG***

(b) **EITHER**

any method from GDC gaining $x \approx 1.32$ ***(M1)(A1)***

maximum value for given domain is 5.11 ***A2***

OR

$\frac{\mathrm{d}A}{\mathrm{d}x} = 8 \cos x - 2$ ***A1***

set $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$, hence $8 \cos x - 2 = 0$ ***M1***

$\cos x = \frac{1}{4} \Rightarrow x \approx 1.32$ ***A1***

hence $A_{\max} = 5.11$ ***A1***

[7 marks]

Examiners report

Generally a well answered question.

An earth satellite moves in a path that can be described by the curve $72.5x^2 + 71.5y^2 = 1$ where $x = x(t)$ and $y = y(t)$ are in thousands of kilometres and t is time in seconds.

Given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 7.75 \times 10^{-5}$ when $x = 3.2 \times 10^{-3}$, find the possible values of $\frac{\mathrm{d}y}{\mathrm{d}t}$.

Give your answers in standard form.

Markscheme

METHOD 1

substituting for x and attempting to solve for y (or vice versa) ***(M1)***

$y = (\pm)0.11821\ldots$ ***(A1)***

EITHER

$145x + 143y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \left(\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{145x}{143y}\right)$ ***M1A1***

OR

$145x\frac{\mathrm{d}x}{\mathrm{d}t} + 143y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ ***M1A1***

THEN

attempting to find $\frac{\mathrm{d}x}{\mathrm{d}t}$ $\left(\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{145(3.2\times10^{-3})}{143((\pm)0.11821\ldots)} \times (7.75 \times 10^{-5})\right)$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13 \times 10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider ±.

METHOD 2

$y = (\pm)\sqrt{\frac{1-72.5x^2}{71.5}}$ ***M1A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)0.0274\ldots$ ***(M1)(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = (\pm)0.0274\ldots \times 7.75 \times 10^{-5}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \pm 2.13 \times 10^{-6}$ ***A1***

Note: Award all marks except the final ***A1*** to candidates who do not consider ±.

[6 marks]

Examiners report

[N/A]

Find the equation of the normal to the curve $y = \frac{\mathrm{e}^x \cos x \ln(x+\mathrm{e})}{(x^{17}+1)^5}$ at the point where $x = 0$.

In your answer give the value of the gradient, of the normal, to three decimal places.

Markscheme

$x = 0 \Rightarrow y = 1$ ***(A1)***

$y'(0) = 1.367879\ldots$ ***(M1)(A1)***

Note: The exact answer is $y'(0) = \frac{\mathrm{e}+1}{\mathrm{e}} = 1 + \frac{1}{\mathrm{e}}$.

so gradient of normal is $\frac{-1}{1.367879\ldots}$ $(= -0.731058\ldots)$ ***(M1)(A1)***

equation of normal is $y = -0.731058\ldots x + c$ ***(M1)***

gives $y = -0.731x + 1$ ***A1***

Note: The exact answer is $y = -\frac{\mathrm{e}}{\mathrm{e}+1}x + 1$.

Accept $y - 1 = -0.731058\ldots(x - 0)$

[7 marks]

Examiners report

Surprisingly many candidates ignored that fact that paper 2 is a calculator paper, attempted an algebraic approach and wasted lots of time.

Candidates that used the GDC were in general successful and achieved 7/7. A number of candidates either found the equation of the tangent or used the positive reciprocal for the normal and many did not find the value of y corresponding to $f(0)$.

Sand is being poured to form a cone of height h cm and base radius r cm. The height remains equal to the base radius at all times. The height of the cone is increasing at a rate of $0.5\ \text{cm min}^{-1}$.

Find the rate at which sand is being poured, in $\text{cm}^3\ \text{min}^{-1}$, when the height is 4 cm.

Markscheme

METHOD 1

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r,\ V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{\text{d}V}{\text{d}h} = \pi h^2$ ***(A1)***

when $h = 4,\ \frac{\text{d}V}{\text{d}t} = \pi \times 4^2 \times 0.5$ (using $\frac{\text{d}V}{\text{d}t} = \frac{\text{d}V}{\text{d}h} \times \frac{\text{d}h}{\text{d}t}$) ***M1A1***

$\frac{\text{d}V}{\text{d}t} = 8\pi\ (= 25.1)\ (\text{cm}^3\ \text{min}^{-1})$ ***A1***

METHOD 2

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r,\ V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{\text{d}V}{\text{d}t} = \frac{1}{3}\pi \times 3h^2 \times \frac{\text{d}h}{\text{d}t}$ ***A1***

when $h = 4,\ \frac{\text{d}V}{\text{d}t} = \pi \times 4^2 \times 0.5$ ***M1A1***

$\frac{\text{d}V}{\text{d}t} = 8\pi\ (= 25.1)\ (\text{cm}^3\ \text{min}^{-1})$ ***A1***

METHOD 3

$V = \frac{1}{3}\pi r^2 h$

$\frac{\text{d}V}{\text{d}t} = \frac{1}{3}\pi\left(2rh\frac{\text{d}r}{\text{d}t} + r^2\frac{\text{d}h}{\text{d}t}\right)$ ***M1A1***

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when $h = 4,\ r = 4,\ \frac{\text{d}V}{\text{d}t} = \frac{1}{3}\pi\left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5\right)$ ***M1A1***

$\frac{\text{d}V}{\text{d}t} = 8\pi\ (= 25.1)\ (\text{cm}^3\ \text{min}^{-1})$ ***A1***

[5 marks]

Examiners report

[N/A]

By using the substitution $x = \sin t$, find $\int \frac{x^3}{\sqrt{1-x^2}}\text{d}x$.

Markscheme

$x = \sin t,\ \mathrm{d}x = \cos t\,\mathrm{d}t$

$\int \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = \int \frac{\sin^3 t}{\sqrt{1-\sin^2 t}}\cos t\,\mathrm{d}t$ ***M1***

$= \int \sin^3 t\,\mathrm{d}t$ ***(A1)***

$= \int \sin^2 t \sin t\,\mathrm{d}t$

$= \int (1-\cos^2 t)\sin t\,\mathrm{d}t$ ***M1A1***

$= \int \sin t\,\mathrm{d}t - \int \cos^2 t \sin t\,\mathrm{d}t$

$= -\cos t + \frac{\cos^3 t}{3} + C$ ***A1A1***

$= -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$ ***A1***

$\left(= -\sqrt{1-x^2}\left(1 - \frac{1}{3}(1-x^2)\right) + C\right)$

$\left(= -\frac{1}{3}\sqrt{1-x^2}(2+x^2) + C\right)$

[7 marks]

Examiners report

Just a few candidates got full marks in this question. Substitution was usually incorrectly done and lead to wrong results. A cosine term in the denominator was a popular error. Candidates often chose unhelpful trigonometric identities and attempted integration by parts. Results such as $\int \sin^3 t\,\mathrm{d}t = \frac{\sin^4 t}{4} + C$ were often seen along with other misconceptions concerning the manipulation/simplification of integrals were also noticed. Some candidates unsatisfactorily attempted to use $\arcsin x$. However, there were some good solutions involving an expression for the cube of $\sin t$ in terms of $\sin t$ and $\sin 3t$. Very few candidates re-expressed their final result in terms of x.

A cone has height h and base radius r . Deduce the formula for the volume of this cone by rotating the triangular region, enclosed by the line $y = h - \frac{h}{r}x$ and the coordinate axes, through 2π about the y-axis.

Markscheme

$x = r - \frac{r}{h}y$ or $x = \frac{r}{h}(h-y)$ (or equivalent) ***(A1)***

$\int \pi x^2 \mathrm{d}y$

$= \pi\int_0^h \left(r - \frac{r}{h}y\right)^2 \mathrm{d}y$ ***M1A1***

Note: Award ***M1*** for $\int x^2 \mathrm{d}y$ and ***A1*** for correct expression.
Accept $\pi\int_0^h \left(\frac{r}{h}y - r\right)^2 \mathrm{d}y$ and $\pi\int_0^h \left(\pm\left(r - \frac{r}{h}x\right)\right)^2 \mathrm{d}x$

$= \pi\int_0^h \left(r^2 - \frac{2r^2}{h}y + \frac{r^2}{h^2}y^2\right)\mathrm{d}y$ ***A1***

Note: Accept substitution method and apply markscheme to corresponding steps.

$= \pi\left[r^2 y - \frac{r^2y^2}{h} + \frac{r^2y^3}{3h^2}\right]_0^h$ ***M1A1***

Note: Award ***M1*** for attempted integration of any quadratic trinomial.

$= \pi \left(r^2h - r^2h + \frac{1}{3}r^2h \right)$ ***M1A1***

Note: Award ***M1*** for attempted substitution of limits in a trinomial.

$= \frac{1}{3}\pi r^2 h$ ***A1***

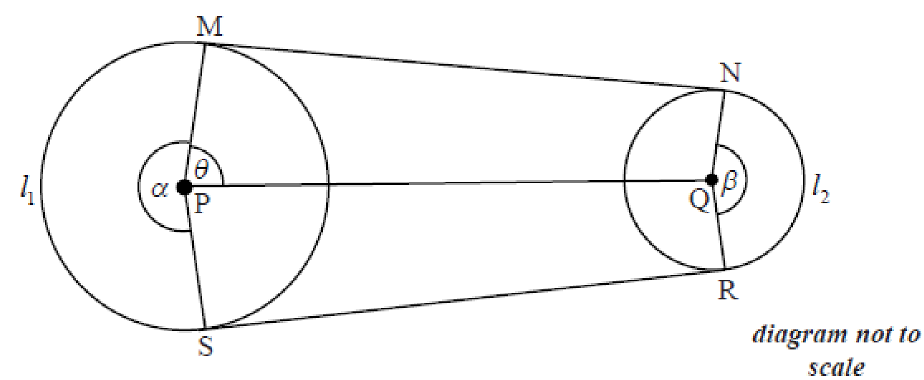
Note: Throughout the question do not penalize missing dx/dy as long as the integrations are done with respect to correct variable.

[9 marks]

Examiners report

Most candidates attempted this question using either the formula given in the information booklet or the disk method. However, many were not successful, either because they started off with the incorrect expression or incorrect integration limits or even attempted to integrate the correct expression with respect to the incorrect variable.

Two non-intersecting circles C_1 , containing points M and S , and C_2 , containing points N and R, have centres P and Q where $PQ = 50$. The line segments [MN] and [SR] are common tangents to the circles. The size of the reflex angle MPS is α, the size of the obtuse angle NQR is β , and the size of the angle MPQ is θ . The arc length MS is l_1 and the arc length NR is l_2 . This information is represented in the diagram below.



The radius of C_1 is x , where $x \geqslant 10$ and the radius of C_2 is 10.

(a) Explain why $x < 40$.

(b) Show that cosθ = x −10 50.

(c) (i) Find an expression for MN in terms of x .

(ii) Find the value of x that maximises MN.

(d) Find an expression in terms of x for

(i) α ;

(ii) β .

(e) The length of the perimeter is given by $l_1 + l_2 + \text{MN} + \text{SR}$.

(i) Find an expression, $b(x)$, for the length of the perimeter in terms of x .

(ii) Find the maximum value of the length of the perimeter.

(iii) Find the value of x that gives a perimeter of length 200.

Markscheme

(a) $\mathrm{PQ} = 50$ and non-intersecting ***R1***

[1 mark]

(b) a construction QT (where T is on the radius MP), parallel to MN, so that $\mathrm{Q\hat{T}M} = 90^\circ$ (angle between tangent and radius $= 90^\circ$) ***M1***

lengths 50, $x - 10$ and angle θ marked on a diagram, or equivalent ***R1***

Note: Other construction lines are possible.

[2 marks]

(c) (i) $\mathrm{MN} = \sqrt{50^2 - (x - 10)^2}$ ***A1***

(ii) maximum for MN occurs when $x = 10$ ***A1***

[2 marks]

(d) (i) $\alpha = 2\pi - 2\theta$ ***M1***

$= 2\pi - 2\arccos\left(\frac{x-10}{50}\right)$ ***A1***

(ii) $\beta = 2\pi - \alpha \quad (= 2\theta\,)$ ***A1***

$= 2\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)$ ***A1***

[4 marks]

(e) (i) $b(x) = x\alpha + 10\beta + 2\sqrt{50^2 - (x - 10)^2}$ ***A1A1A1***

$= x\left(2\pi - 2\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)\right) + 20\left(\left(\cos^{-1}\left(\frac{x-10}{50}\right)\right)\right) + 2\sqrt{50^2 - (x - 10)^2}$ ***M1A1***

(ii) maximum value of perimeter $= 276$ ***A2***

(iii) perimeter of 200 cm $b(x) = 200$ ***(M1)***

when $x = 21.2$ ***A1***

[9 marks]

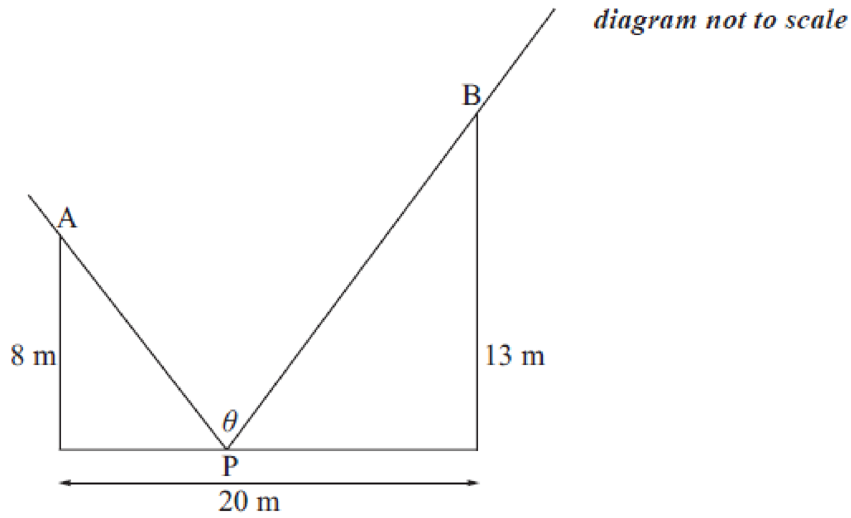
Total [18 marks]

Examiners report

This is not an inherently difficult question, but candidates either made heavy weather of it or avoided it almost entirely. The key to answering the question is in obtaining the displayed answer to part (b), for which a construction line parallel to MN through Q is required. Diagrams seen by examiners on some scripts tend to suggest that the perpendicularity property of a tangent to a circle and the associated radius is not as firmly known as they had expected. Some candidates mixed radians and degrees in their expressions.

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres.

The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \hat{\mathrm{APB}}$, as shown in the diagram.



a. Find an expression for θ in terms of x, where x is the distance of P from the base of the wall of height 8 m. [2]

b. (i) Calculate the value of θ when $x = 0$. [2]

(ii) Calculate the value of θ when $x = 20$.

c. Sketch the graph of θ, for $0 \leqslant x \leqslant 20$. [2]

d. Show that $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$. [6]

e. Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. [3]

f. The point P moves across the street with speed $0.5\ \mathrm{ms}^{-1}$. Determine the rate of change of θ with respect to time when P is at the midpoint of the street. [4]

Markscheme

a. **EITHER**

$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent) ***M1A1***

Note: Accept $\theta = 180^\circ - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$ (or equivalent).

OR

$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$ (or equivalent) ***M1A1***

[2 marks]

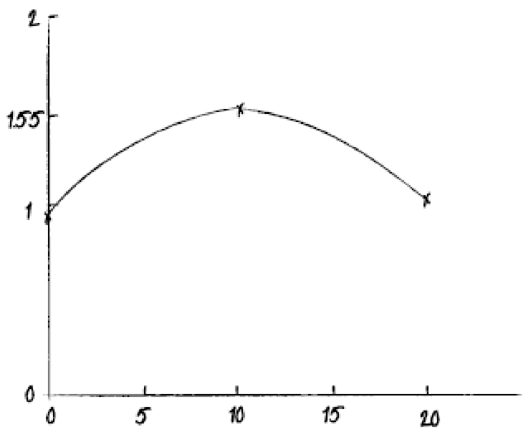
b. (i) $\theta = 0.994$ $\left(= \arctan\frac{20}{13}\right)$ ***A1***

(ii) $\theta = 1.19$ $\left(= \arctan\frac{5}{2}\right)$ ***A1***

[2 marks]

c. correct shape. ***A1***

correct domain indicated. ***A1***



[2 marks]

d. attempting to differentiate one $\arctan(f(x))$ term ***M1***

EITHER

$\theta = \pi - \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{13}{20-x}\right)$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{8}{x^2} \times \frac{1}{1+\left(\frac{8}{x}\right)^2} - \frac{13}{(20-x)^2} \times \frac{1}{1+\left(\frac{13}{20-x}\right)^2}$ ***A1A1***

OR

$\theta = \arctan\left(\frac{x}{8}\right) + \arctan\left(\frac{20-x}{13}\right)$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\frac{1}{8}}{1+\left(\frac{x}{8}\right)^2} + \frac{-\frac{1}{13}}{1+\left(\frac{20-x}{13}\right)^2}$ ***A1A1***

THEN

$= \frac{8}{x^2+64} - \frac{13}{569-40x+x^2}$ ***A1***

$= \frac{8(569-40x+x^2)-13(x^2+64)}{(x^2+64)(x^2-40x+569)}$ ***M1A1***

$= \frac{5(744-64x-x^2)}{(x^2+64)(x^2-40x+569)}$ ***AG***

[6 marks]

e. Maximum light intensity at P occurs when $\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$. ***(M1)***

either attempting to solve $\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$ for x or using the graph of either θ or $\frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$x = 10.05$ (m) ***A1***

[3 marks]

f. $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5$ ***(A1)***

At $x = 10$, $\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0.000453 \left(= \frac{5}{11029}\right)$. ***(A1)***

use of $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.000227 \left(= \frac{5}{22058}\right)$ (rad s^{-1}) ***A1***

Note: Award ***(A1)*** for $\frac{\mathrm{d}x}{\mathrm{d}t} = -0.5$ and ***A1*** for $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -0.000227 \left(= -\frac{5}{22058}\right)$.

Note: Implicit differentiation can be used to find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$. Award as above.

[4 marks]

Examiners report

a. Part (a) was reasonably well done. While many candidates exhibited sound trigonometric knowledge to correctly express θ in terms of x, many other candidates were not able to use elementary trigonometry to formulate the required expression for θ.

b. In part (b), a large number of candidates did not realize that θ could only be acute and gave obtuse angle values for θ. Many candidates also demonstrated a lack of insight when substituting endpoint x-values into θ.

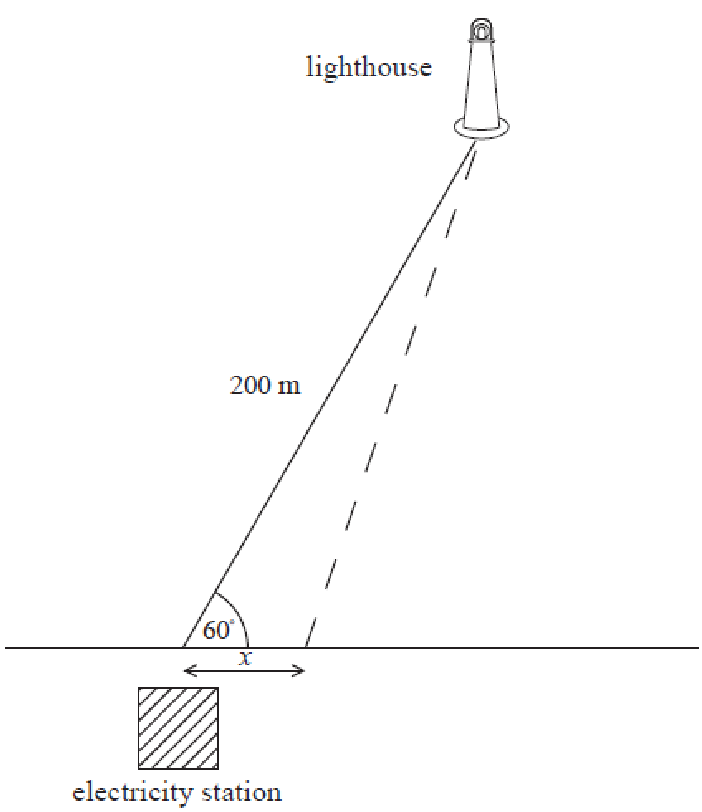
c. In part (c), many candidates sketched either inaccurate or implausible graphs.

d. In part (d), a large number of candidates started their differentiation incorrectly by failing to use the chain rule correctly.

e. For a question part situated at the end of the paper, part (e) was reasonably well done. A large number of candidates demonstrated a sound knowledge of finding where the maximum value of θ occurred and rejected solutions that were not physically feasible.

f. In part (f), many candidates were able to link the required rates, however only a few candidates were able to successfully apply the chain rule in a related rates context.

An electricity station is on the edge of a straight coastline. A lighthouse is located in the sea 200 m from the electricity station. The angle between the coastline and the line joining the lighthouse with the electricity station is 60°. A cable needs to be laid connecting the lighthouse to the electricity station. It is decided to lay the cable in a straight line to the coast and then along the coast to the electricity station. The length of cable laid along the coastline is x metres. This information is illustrated in the diagram below.



The cost of laying the cable along the sea bed is US$80 per metre, and the cost of laying it on land is US$20 per metre.

a. Find, in terms of x, an expression for the cost of laying the cable. [4]

b. Find the value of x, to the nearest metre, such that this cost is minimized. [2]

Markscheme

a. let the distance the cable is laid along the seabed be y

$y^2 = x^2 + 200^2 - 2 \times x \times 200 \cos 60°$ ***(M1)***

(or equivalent method)

$y^2 = x^2 - 200x + 40000$ ***(A1)***

cost = $C = 80y + 20x$ ***(M1)***

$C = 80(x^2 - 200x + 40000)^{\frac{1}{2}} + 20x$ ***A1***

[4 marks]

b. $x = 55.2786\ldots = 55$ (m to the nearest metre) ***(A1)A1***

$\left(x = 100 - \sqrt{2000}\right)$

[2 marks]

Examiners report

a. Some surprising misconceptions were evident here, using right angled trigonometry in non right angled triangles etc. Those that used the cosine rule, usually managed to obtain the correct answer to part (a).

b. Some surprising misconceptions were evident here, using right angled trigonometry in non right angled triangles etc. Many students attempted to find the value of the minimum algebraically instead of the simple calculator solution.

---

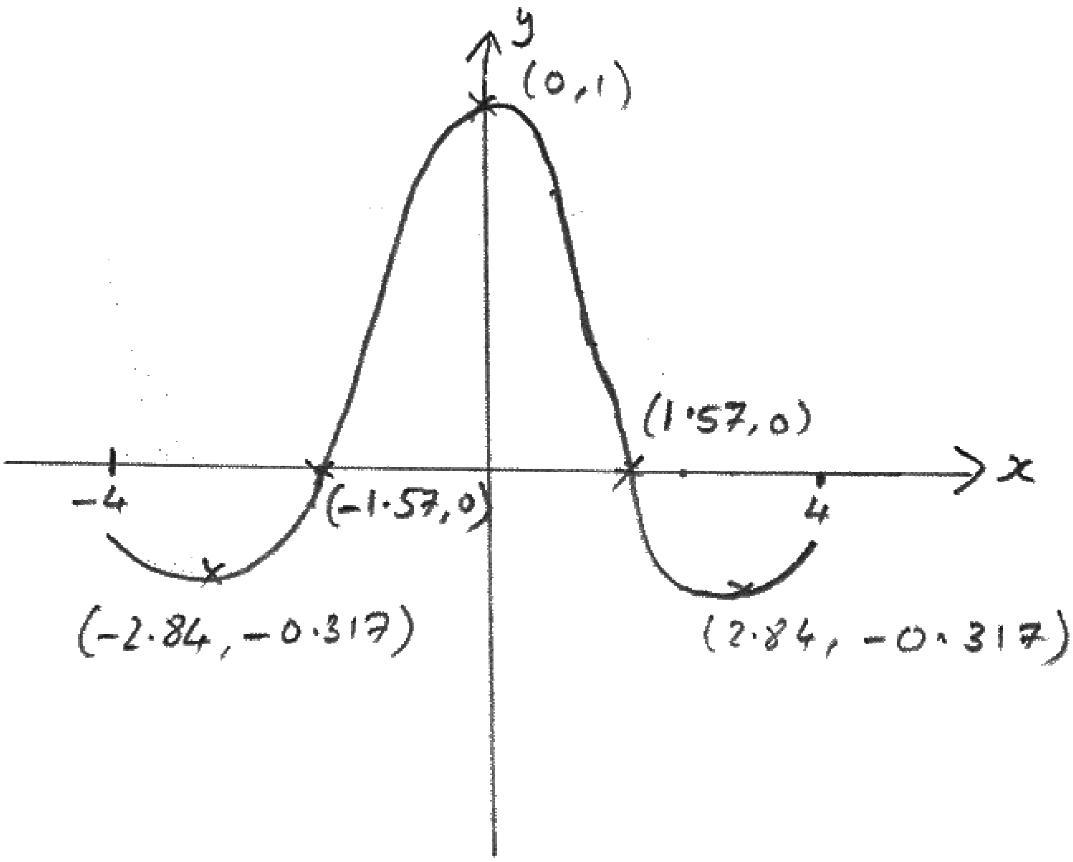
a. Sketch the curve $y = \frac{\cos x}{\sqrt{x^2+1}},\ -4 \leqslant x \leqslant 4$ showing clearly the coordinates of the x-intercepts, any maximum points and any minimum points. [4]

b. Write down the gradient of the curve at $x = 1$. [1]

c. Find the equation of the normal to the curve at $x = 1$. [3]

Markscheme

a.



A1A1A1A1

Note: Award ***A1*** for correct shape. Do not penalise if too large a domain is used,
A1 for correct x-intercepts,
A1 for correct coordinates of two minimum points,
A1 for correct coordinates of maximum point.
Accept answers which correctly indicate the position of the intercepts, maximum point and minimum points.

[4 marks]

b. gradient at $x = 1$ is -0.786 ***A1***

[1 mark]

c. gradient of normal is $\frac{-1}{-0.786}$ $(= 1.272...)$ ***(A1)***

when $x = 1$, $y = 0.3820...$ ***(A1)***
Equation of normal is $y - 0.382 = 1.27(x - 1)$ ***A1***
$(\Rightarrow y = 1.27x - 0.890)$
[3 marks]

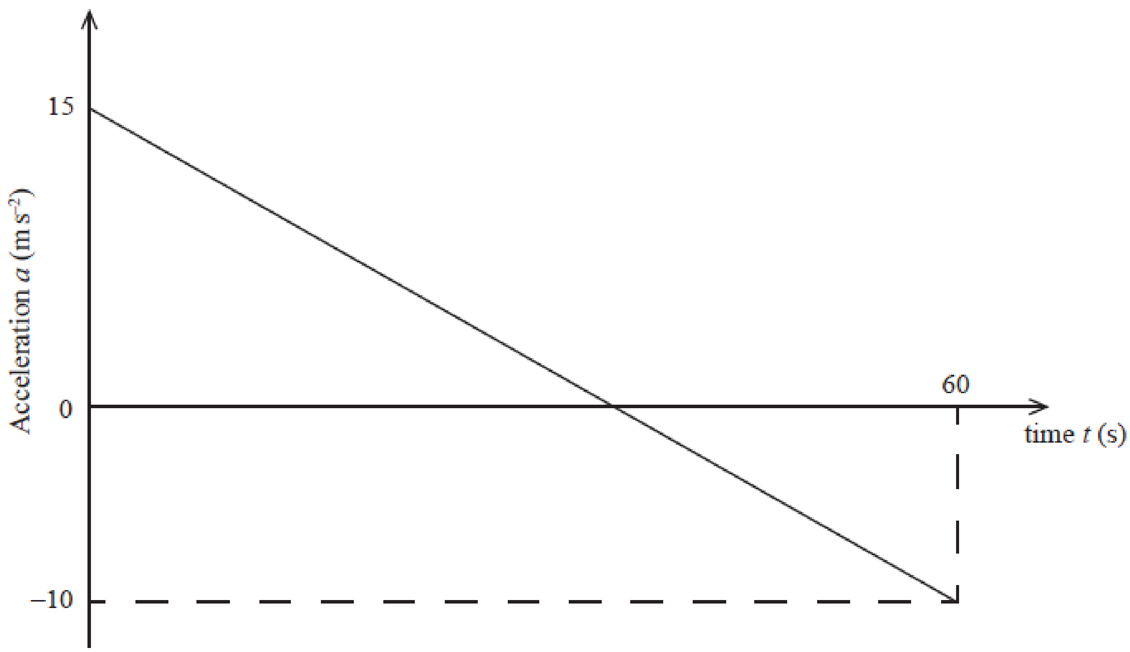
Examiners report

a. Most candidates were able to make a meaningful start to this question, but many made errors along the way and hence only a relatively small number of candidates gained full marks for the question. Common errors included trying to use degrees, rather than radians, trying to use algebraic methods to find the gradient in part (b) and trying to find the equation of the tangent rather than the equation of the normal in part (c).

b. Most candidates were able to make a meaningful start to this question, but many made errors along the way and hence only a relatively small number of candidates gained full marks for the question. Common errors included trying to use degrees, rather than radians, trying to use algebraic methods to find the gradient in part (b) and trying to find the equation of the tangent rather than the equation of the normal in part (c).

c. Most candidates were able to make a meaningful start to this question, but many made errors along the way and hence only a relatively small number of candidates gained full marks for the question. Common errors included trying to use degrees, rather than radians, trying to use algebraic methods to find the gradient in part (b) and trying to find the equation of the tangent rather than the equation of the normal in part (c).

A jet plane travels horizontally along a straight path for one minute, starting at time $t = 0$, where t is measured in seconds. The acceleration, a , measured in ms^{-2}, of the jet plane is given by the straight line graph below.



a. Find an expression for the acceleration of the jet plane during this time, in terms of t . [1]

b. Given that when $t = 0$ the jet plane is travelling at 125 ms^{-1}, find its maximum velocity in ms^{-1} during the minute that follows. [4]

c. Given that the jet plane breaks the sound barrier at 295 ms^{-1}, find out for how long the jet plane is travelling greater than this speed. [3]

Markscheme

a. equation of line in graph $a = -\frac{25}{60}t + 15$ ***A1***

$\left(a = -\frac{5}{12}t + 15\right)$

[1 mark]

b. $\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{5}{12}t + 15$ ***(M1)***

$v = -\frac{5}{24}t^2 + 15t + c$ ***(A1)***

when $t = 0$, $v = 125\ \text{ms}^{-1}$

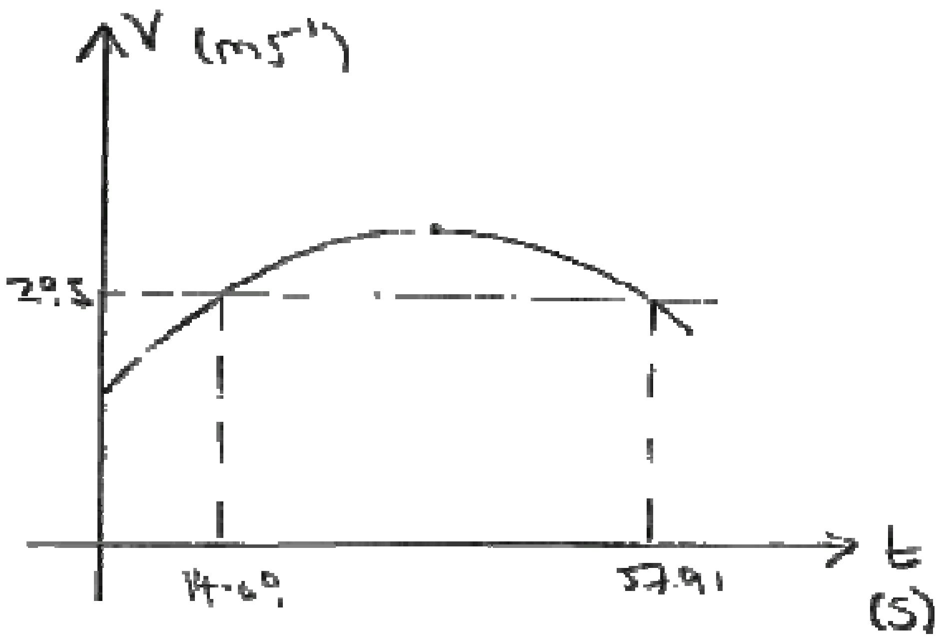
$v = -\frac{5}{24}t^2 + 15t + 125$ ***A1***

from graph or by finding time when $a = 0$

maximum $= 395\ \text{ms}^{-1}$ ***A1***

[4 marks]

c. **EITHER**



graph drawn and intersection with $v = 295\ \text{ms}^{-1}$ ***(M1)(A1)***

\(t = 57.91 - 14.09 = 43.8\) ***A1***

OR

$295 = -\frac{5}{24}t^2 + 15t + 125 \Rightarrow t = 57.91...; 14.09...$

$t = 57.91... - 14.09... = 43.8\ \left(8\sqrt{30}\right)$ ***A1***

[3 marks]

Examiners report

a. This question was well answered by a large number of candidates and indicated a good understanding of calculus, kinematics and use of the graphing calculator. Some candidates worked in x and y rather than a, v and t but mostly obtained correct solutions. Although the majority of candidate used integration throughout the question some correct solutions were obtained by considering the areas in the diagram.

b. This question was well answered by a large number of candidates and indicated a good understanding of calculus, kinematics and use of the graphing calculator. Some candidates worked in x and y rather than a, v and t but mostly obtained correct solutions. Although the majority of candidate used integration throughout the question some correct solutions were obtained by considering the areas in the diagram.

c. This question was well answered by a large number of candidates and indicated a good understanding of calculus, kinematics and use of the graphing calculator. Some candidates worked in x and y rather than a, v and t but mostly obtained correct solutions. Although the majority of candidate used integration throughout the question some correct solutions were obtained by considering the areas in the diagram.

A particle, A, is moving along a straight line. The velocity, v_A ms^{-1}, of A t seconds after its motion begins is given by

$$v_A = t^3 - 5t^2 + 6t.$$

a. Sketch the graph of $v_A = t^3 - 5t^2 + 6t$ for $t \geqslant 0$, with v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t-axis. [3]

b. Write down the times for which the velocity of the particle is increasing. [2]

c. Write down the times for which the magnitude of the velocity of the particle is increasing. [3]

d. At $t = 0$ the particle is at point O on the line. [3]

Find an expression for the particle's displacement, x_Am, from O at time t.

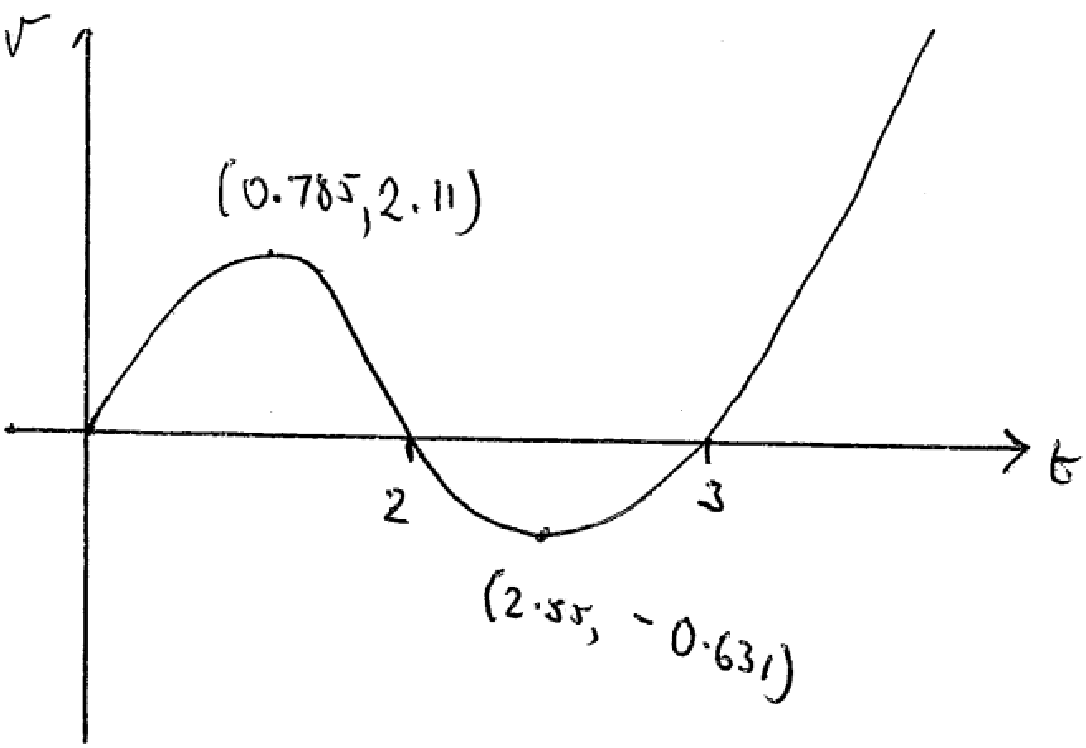
e. A second particle, B, moving along the same line, has position x_B m, velocity v_B ms^{-1} and acceleration, a_B ms^{-2}, where $a_B = -2v_B$ for $t \geqslant 0$. At $t = 0$, $x_B = 20$ and $v_B = -20$. [4]

Find an expression for v_B in terms of t.

f. Find the value of t when the two particles meet. [6]

Markscheme

a.



A1A1A1

Note: Award ***A1*** for general shape, ***A1*** for correct maximum and minimum, ***A1*** for intercepts.

Note: Follow through applies to (b) and (c).

[3 marks]

b. $0 \leqslant t < 0.785$, $\left(\text{or } 0 \leqslant t < \frac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

and $t > 2.55$ $\left(\text{or } t > \frac{5+\sqrt{7}}{3}\right)$ ***A1***

[2 marks]

c. $0 \leqslant t < 0.785, \left(\text{or } 0 \leqslant t < \frac{5-\sqrt{7}}{3}\right)$ ***A1***

(allow $t < 0.785$)

$2 < t < 2.55, \left(\text{or } 2 < t < \frac{5+\sqrt{7}}{3}\right)$ ***A1***

$t > 3$ ***A1***

[3 marks]

d. position of A: $x_A = \int t^3 - 5t^2 + 6t \, \mathrm{d}t$ ***(M1)***

$x_A = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 \quad (+c)$ ***A1***

when $t = 0, \ x_A = 0$, so $c = 0$ ***R1***

[3 marks]

e. $\frac{\mathrm{d}v_B}{\mathrm{d}t} = -2v_B \Rightarrow \int \frac{1}{v_B} \mathrm{d}v_B = \int -2\mathrm{d}t$ ***(M1)***

$\ln|v_B| = -2t + c$ ***(A1)***

$v_B = Ae^{-2t}$ ***(M1)***

$v_B = -20$ when $t = 0$ so $v_B = -20e^{-2t}$ ***A1***

[4 marks]

f. $x_B = 10e^{-2t}(+c)$ ***(M1)(A1)***

$x_B = 20$ when $t = 0$ so $x_B = 10e^{-2t} + 10$ ***(M1)A1***

meet when $\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 = 10e^{-2t} + 10$ ***(M1)***

$t = 4.41(290\ldots)$ ***A1***

[6 marks]

Examiners report

a. Part (a) was generally well done, although correct accuracy was often a problem.

b. Parts (b) and (c) were also generally quite well done.

c. Parts (b) and (c) were also generally quite well done.

d. A variety of approaches were seen in part (d) and many candidates were able to obtain at least 2 out of 3. A number missed to consider the $+c$, thereby losing the last mark.

e. Surprisingly few candidates were able to solve part (e) correctly. Very few could recognise the easy variable separable differential equation. As a consequence part (f) was frequently left.

f. Surprisingly few candidates were able to solve part (e) correctly. Very few could recognise the easy variable separable differential equation. As a consequence part (f) was frequently left.

The function f is defined by

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}, \text{ for } x \in D,$$

where $D \subseteq \mathbb{R}$ is the greatest possible domain of f.

(a) Find the roots of $f(x) = 0$.

(b) Hence specify the set D.

(c) Find the coordinates of the local maximum on the graph $y = f(x)$.

(d) Solve the equation $f(x) = 3$.

(e) Sketch the graph of $|y| = f(x)$, for $x \in D$.

(f) Find the area of the region completely enclosed by the graph of $|y| = f(x)$

Markscheme

(a) solving to obtain one root: 1, – 2 or – 5 ***A1***

obtain other roots ***A1***

[2 marks]

(b) $D = x \in [-5,\ -2] \cup [1,\ \infty)$ (or equivalent) ***M1A1***

Note: ***M1*** is for 1 finite and 1 infinite interval.

[2 marks]

(c) coordinates of local maximum $-3.73 - 2 - \sqrt{3},\ 3.22\sqrt{6\sqrt{3}}$ ***A1A1***

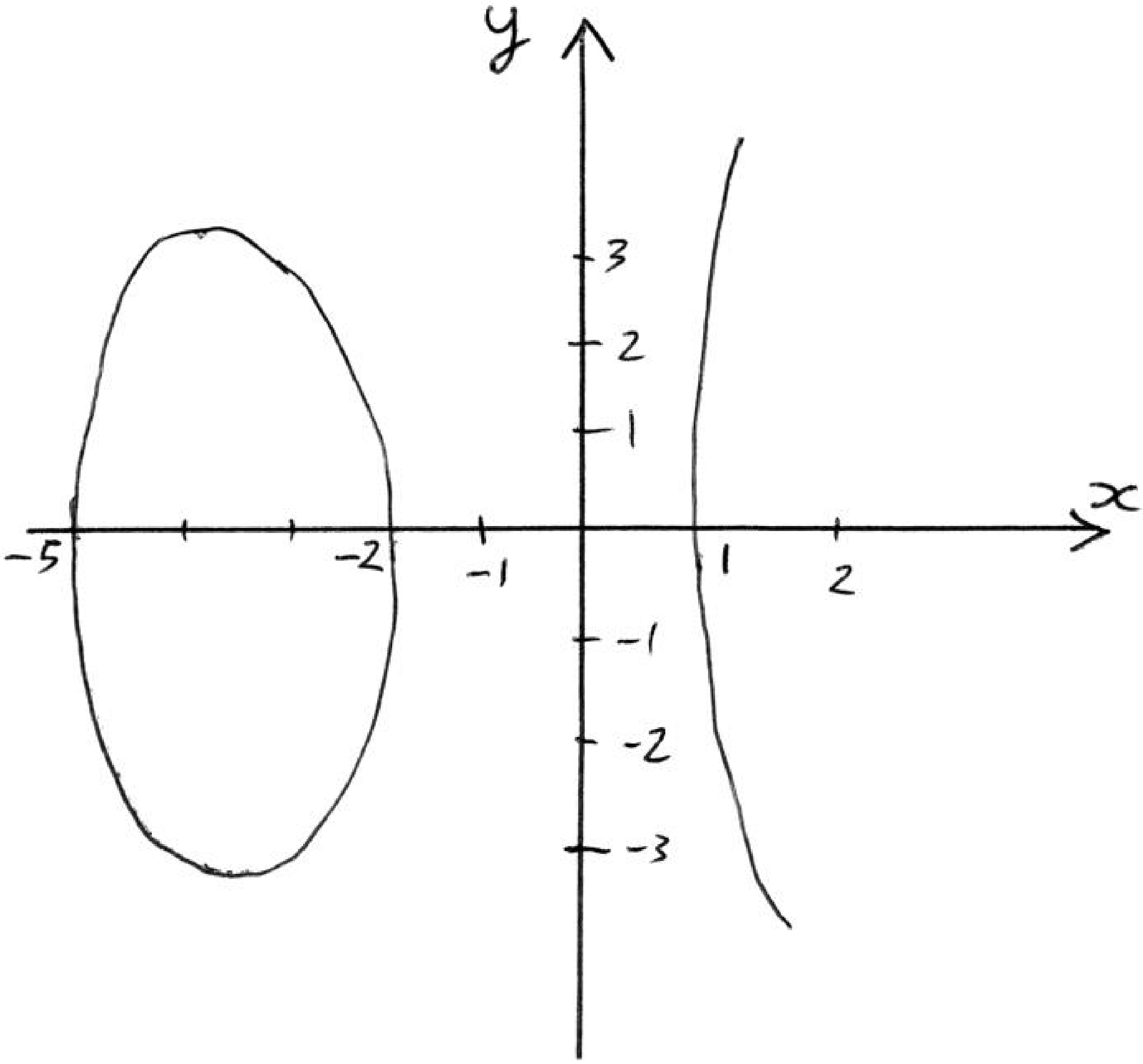
[2 marks]

(d) use GDC to obtain one root: 1.41, – 3.18 or – 4.23 ***A1***

obtain other roots ***A1***

[2 marks]

(e)



A1A1A1

Note: Award ***A1*** for shape, ***A1*** for max and for min clearly in correct places, ***A1*** for all intercepts.

Award ***A1A0A0*** if only the complete top half is shown.

[3 marks]

(f) required area is twice that of $y = f(x)$ between – 5 and – 2 ***M1A1***

answer 14.9 ***A1*** ***N3***

Note: Award ***M1A0A0*** for $\int_{-5}^{-2} f(x)\mathrm{d}x = 7.47\ldots$ or ***N1*** for 7.47.

[3 marks]

Total [14 marks]

Examiners report

This was a multi-part question that was well answered by many candidates. The main difficulty was sketching the graph and this meant that the last part was not well answered.

The function f is defined on the domain [0, 2] by $f(x) = \ln(x+1)\sin(\pi x)$.

a. Obtain an expression for $f'(x)$. [3]

b. Sketch the graphs of f and f' on the same axes, showing clearly all x-intercepts. [4]

c. Find the x-coordinates of the two points of inflexion on the graph of f. [2]

d. Find the equation of the normal to the graph of f where $x = 0.75$, giving your answer in the form $y = mx + c$. [3]

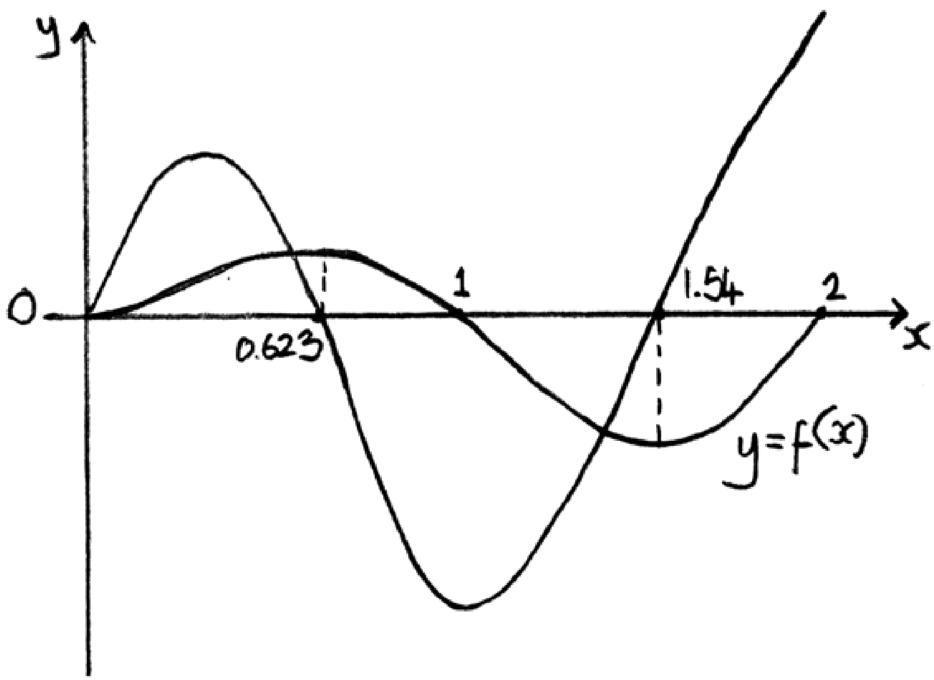
e. Consider the points A $(a\ ,\ f(a))$, B $(b\ ,\ f(b))$ and C $(c\ ,\ f(c))$ where a , b and c $(a < b < c)$ are the solutions of the equation $f(x) = f'(x)$. Find the area of the triangle ABC. [6]

Markscheme

a. $f'(x) = \frac{1}{x+1}\sin(\pi x) + \pi\ln(x+1)\cos(\pi x)$ ***M1A1A1***

[3 marks]

b.



A4

Note: Award ***A1A1*** for graphs, ***A1A1*** for intercepts.

[4 marks]

c. 0.310, 1.12 ***A1A1***

[2 marks]

d. $f'(0.75) = -0.839092$ ***A1***

so equation of normal is $y - 0.39570812 = \frac{1}{0.839092}(x - 0.75)$ ***M1***

$y = 1.19x - 0.498$ ***A1***

[3 marks]

e. $\mathrm{A}(0,\ 0)$

$\mathrm{B}(\overbrace{0.548\ldots}^{c}, \overbrace{0.432\ldots}^{d})$ ***A1***

$\mathrm{C}(\overbrace{1.44\ldots}^{e}, \overbrace{-0.881\ldots}^{f})$ ***A1***

Note: Accept coordinates for B and C rounded to 3 significant figures.

area $\Delta\mathrm{ABC} = \frac{1}{2}|(c\boldsymbol{i} + d\boldsymbol{j}) \times (e\boldsymbol{i} + f\boldsymbol{j})|$ ***M1A1***

$= \frac{1}{2}(de - cf)$ ***A1***

$= 0.554$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

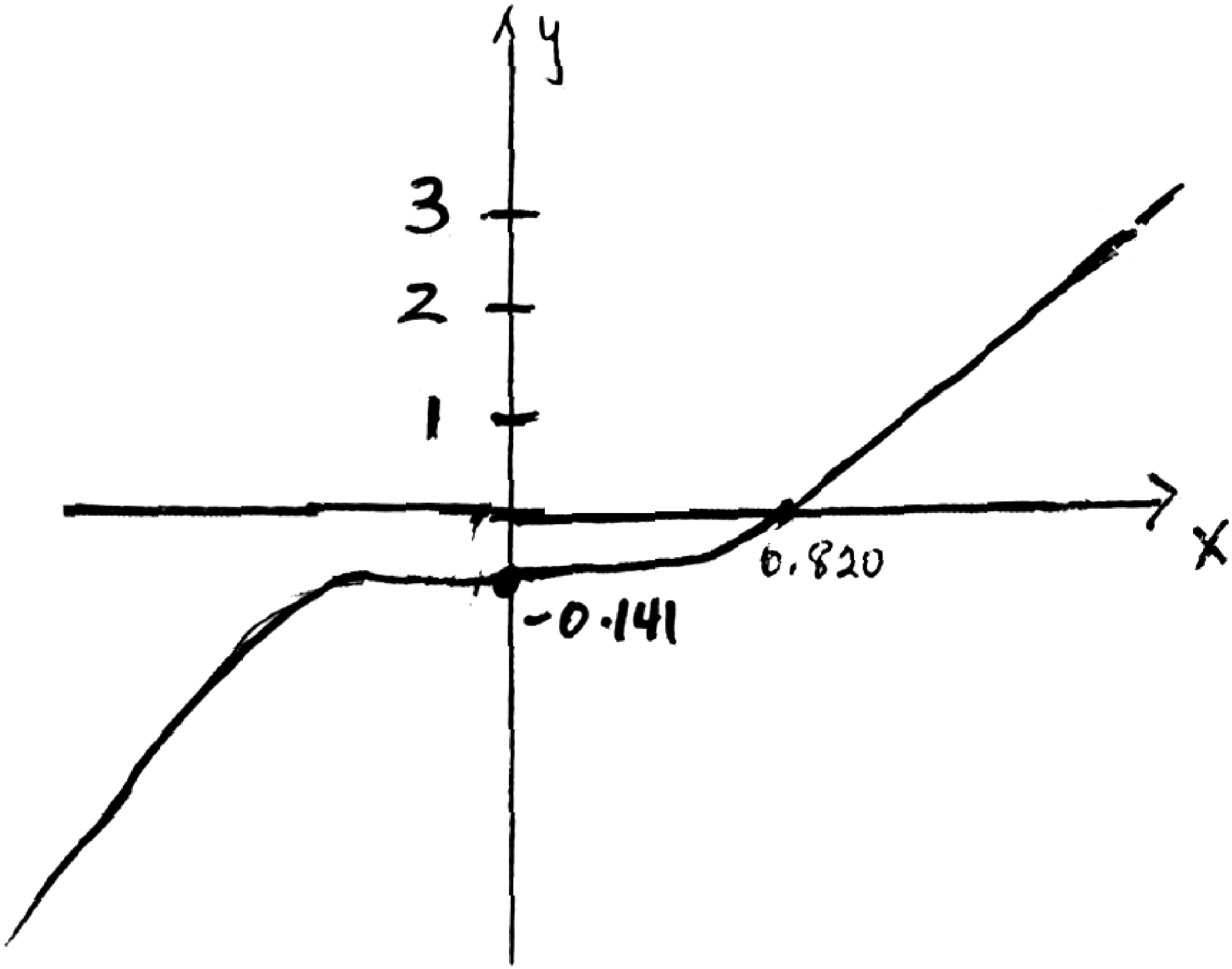
Consider the graph of $y = x + \sin(x - 3), \ -\pi \leqslant x \leqslant \pi$.

a. Sketch the graph, clearly labelling the *x* and *y* intercepts with their values. [3]

b. Find the area of the region bounded by the graph and the *x* and *y* axes. [2]

Markscheme

a.



A1A1A1

Note: Award ***A1*** for shape,

A1 for *x*-intercept is 0.820, accept $\sin(-3)$ or $-\sin(3)$

A1 for *y*-intercept is −0.141.

[3 marks]

b. $A = \int_0^{0.8202} |x + \sin(x - 3)| \, \mathrm{d}x \approx 0.0816$ sq units ***(M1)A1***

[2 marks]

Examiners report

a. Many candidates attempted this question successfully. In (a), however, a large number of candidates did not use the zoom feature of the GDC to draw an accurate sketch of the given function. In (b), some candidates used the domain as the limits of the integral. Other candidates did not take the absolute value of the integral.

b. Many candidates attempted this question successfully. In (a), however, a large number of candidates did not use the zoom feature of the GDC to draw an accurate sketch of the given function. In (b), some candidates used the domain as the limits of the integral. Other candidates did not take the absolute value of the integral.

Consider the curve with equation $f(x) = \mathrm{e}^{-2x^2}$ for $x < 0$.

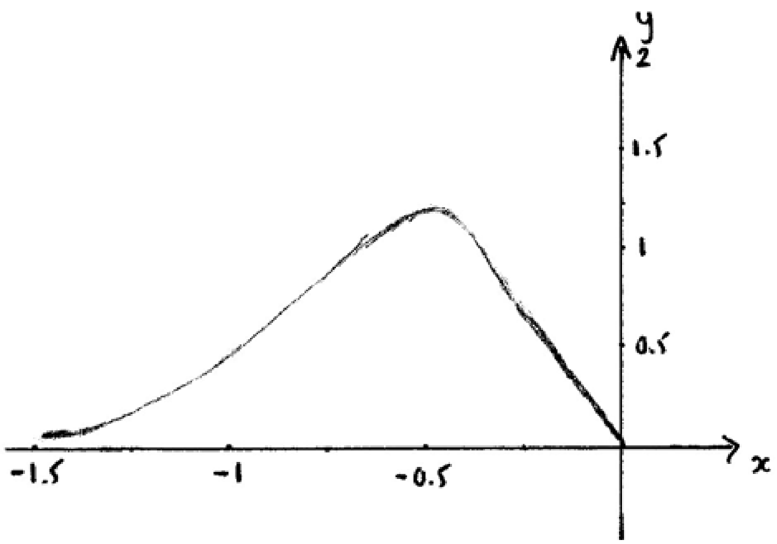
Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

Markscheme

METHOD 1

EITHER

Using the graph of $y = f'(x)$ ***(M1)***

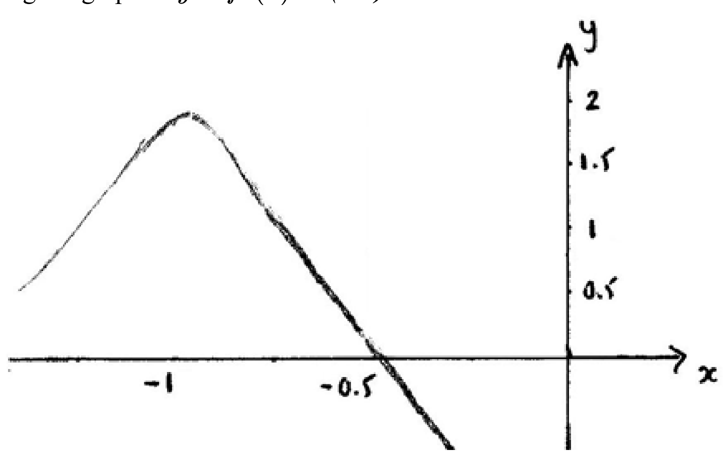


A1

The maximum of $f'(x)$ occurs at $x = -0.5$. ***A1***

OR

Using the graph of $y = f''(x)$. ***(M1)***



A1

The zero of $f''(x)$ occurs at $x = -0.5$. ***A1***

THEN

Note: Do not award this ***A1*** for stating $x = \pm 0.5$ as the final answer for x .

$f(-0.5) = 0.607 (= \mathrm{e}^{-0.5})$ ***A2***

Note: Do not award this ***A1*** for also stating (0.5, 0.607) as a coordinate.

EITHER

Correctly labelled graph of $f'(x)$ for $x < 0$ denoting the maximum $f'(x)$ ***R1***

(*e.g.* $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) ***A1*** ***N2***

OR

Correctly labelled graph of $f''(x)$ for $x < 0$ denoting the maximum $f'(x)$ ***R1***

(*e.g.* $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) ***A1*** ***N2***

OR

$f'(0.5) \approx 1.21$. $f'(x) < 1.21$ just to the left of $x = -\frac{1}{2}$

and $f'(x) < 1.21$ just to the right of $x = -\frac{1}{2}$ ***R1***

(*e.g.* $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) ***A1*** ***N2***

OR

$f''(x) > 0$ just to the left of $x = -\frac{1}{2}$ and $f''(x) < 0$ just to the right of $x = -\frac{1}{2}$ ***R1***

(*e.g.* $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) ***A1*** ***N2***

[7 marks]

METHOD 2

$f'(x) = -4x\mathrm{e}^{-2x^2}$ ***A1***

$f''(x) = -4\mathrm{e}^{-2x^2} + 16x^2\mathrm{e}^{-2x^2} \quad \left(= (16x^2 - 4)\mathrm{e}^{-2x^2}\right)$ ***A1***

Attempting to solve $f''(x) = 0$ ***(M1)***

$x = -\frac{1}{2}$ ***A1***

Note: Do not award this ***A1*** for stating $x = \pm\frac{1}{2}$ as the final answer for x .

$f\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{\mathrm{e}}} \ (= 0.607)$ ***A1***

Note: Do not award this ***A1*** for also stating $\left(\frac{1}{2}, \frac{1}{\sqrt{\mathrm{e}}}\right)$ as a coordinate.

EITHER

Correctly labelled graph of $f'(x)$ for $x < 0$ denoting the maximum $f'(x)$ ***R1***

(*e.g.* $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated) ***A1*** ***N2***

OR

Correctly labelled graph of $f''(x)$ for $x < 0$ denoting the maximum $f'(x)$ ***R1***

(*e.g.* $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated) ***A1*** ***N2***

OR

$f'(0.5) \approx 1.21$. $f'(x) < 1.21$ just to the left of $x = -\frac{1}{2}$

and $f'(x) < 1.21$ just to the right of $x = -\frac{1}{2}$ ***R1***

($e.g.$ $f'(-0.6) = 1.17$ and $f'(-0.4) = 1.16$ stated)   ***A1***   ***N2***

OR

$f''(x) > 0$ just to the left of $x = -\frac{1}{2}$ and $f''(x) < 0$ just to the right of $x = -\frac{1}{2}$ ***R1***

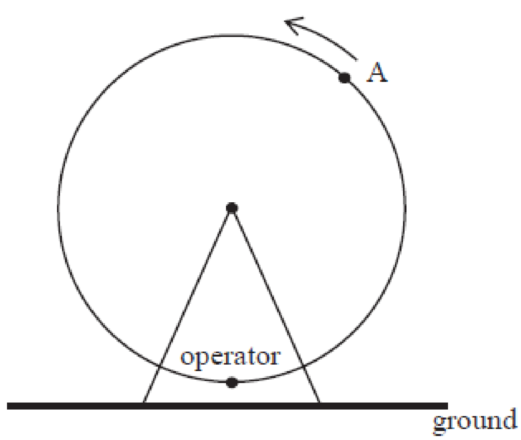
($e.g.$ $f''(-0.6) = 0.857$ and $f''(-0.4) = -1.05$ stated)   ***A1***   ***N2***

[7 marks]

Examiners report

Most candidates adopted an algebraic approach rather than a graphical approach. Most candidates found $f'(x)$ correctly, however when attempting to find $f''(x)$, a surprisingly large number either made algebraic errors using the product rule or seemingly used an incorrect form of the product rule. A large number ignored the domain restriction and either expressed $x = \pm\frac{1}{2}$ as the x-coordinates of the point of inflection or identified $x = \frac{1}{2}$ rather than $x = -\frac{1}{2}$. Most candidates were unsuccessful in their attempts to justify the existence of the point of inflection.

Below is a sketch of a Ferris wheel, an amusement park device carrying passengers around the rim of the wheel.

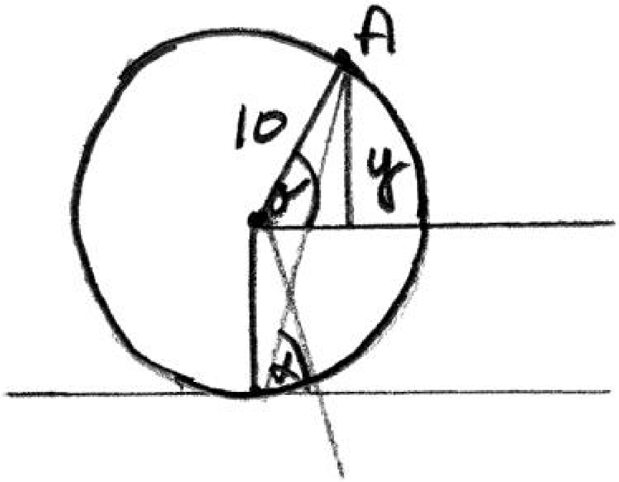


(a) The circular Ferris wheel has a radius of 10 metres and is revolving at a rate of 3 radians per minute. Determine how fast a passenger on the wheel is going vertically upwards when the passenger is at point A, 6 metres higher than the centre of the wheel, and is rising.

(b) The operator of the Ferris wheel stands directly below the centre such that the bottom of the Ferris wheel is level with his eyeline. As he watches the passenger his line of sight makes an angle α with the horizontal. Find the rate of change of α at point A.

Markscheme

(a)



$\frac{d\theta}{dt} = 3$ ***(A1)***

$y = 10\sin\theta$ ***A1***

$\frac{dy}{d\theta} = 10\cos\theta$ ***M1***

$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = 30\cos\theta$ ***M1***

at $y = 6$, $\cos\theta = \frac{8}{10}$ ***(M1)(A1)***

$\Rightarrow \frac{dy}{dt} = 24$ (metres per minute) (accept 24.0) ***A1***

[7 marks]

(b) $\alpha = \frac{\theta}{2} + \frac{\pi}{4}$ ***M1A1***

$\frac{d\alpha}{dt} = \frac{1}{2}\frac{d\theta}{dt} = 1.5$ ***A1***

[3 marks]

Total [10 marks]

Examiners report

Many students were unable to get started with this question, and those that did were generally very poor at defining their variables at the start.

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \leqslant t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{dv}{dt} + v^2 + 1 = 0$.
It is also given that $v = 1$ when $t = 0$.

a. Find an expression for v in terms of t . [7]

b. Sketch the graph of v against t , clearly showing the coordinates of any intercepts, and the equations of any asymptotes. [3]

c. (i) Write down the time T at which the velocity is zero. [3]

(ii) Find the distance travelled in the interval $[0, T]$.

d. Find an expression for s , the displacement, in terms of t , given that $s = 0$ when $t = 0$. [5]

e. Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. [4]

Markscheme

a. $\frac{\mathrm{d}v}{\mathrm{d}t} = -v^2 - 1$

attempt to separate the variables ***M1***

$\int \frac{1}{1+v^2}\mathrm{d}v = \int -1\mathrm{d}t$ ***A1***

$\arctan v = -t + k$ ***A1A1***

Note: Do not penalize the lack of constant at this stage.

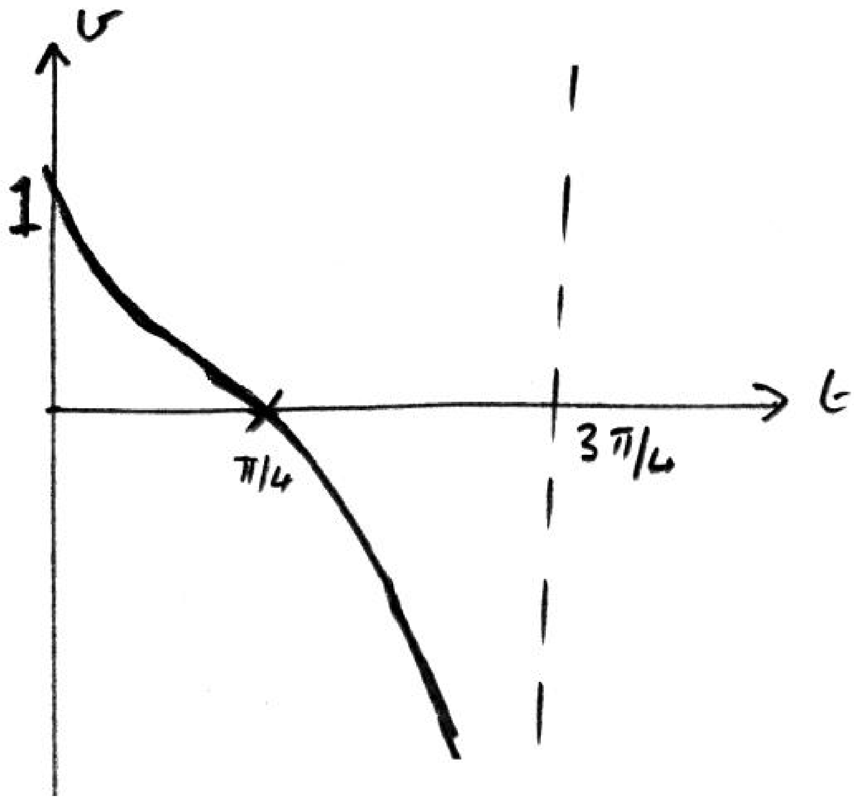
when $t = 0$, $v = 1$ ***M1***

$\Rightarrow k = \arctan 1 = \left(\frac{\pi}{4}\right) = (45^\circ)$ ***A1***

$\Rightarrow v = \tan\left(\frac{\pi}{4} - t\right)$ ***A1***

[7 marks]

b.



A1A1A1

Note: Award ***A1*** for general shape,
A1 for asymptote,
A1 for correct t and v intercept.

Note: Do not penalise if a larger domain is used.

[3 marks]

c. (i) $T = \frac{\pi}{4}$ ***A1***

(ii) area under curve $= \int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - t\right)\mathrm{d}t$ ***(M1)***

$= 0.347 \left(= \frac{1}{2}\ln 2\right)$ ***A1***

[3 marks]

d. $v = \tan\left(\frac{\pi}{4} - t\right)$

$s = \int \tan\left(\frac{\pi}{4} - t\right)\mathrm{d}t$ ***M1***

$\int \frac{\sin\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)}\,\mathrm{d}t$ ***(M1)***

$= \ln\cos\left(\frac{\pi}{4} - t\right) + k$ ***A1***

when $t = 0,\ s = 0$

$k = -\ln\cos\frac{\pi}{4}$ ***A1***

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4} \left(= \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - t\right)\right]\right)$ ***A1***

[5 marks]

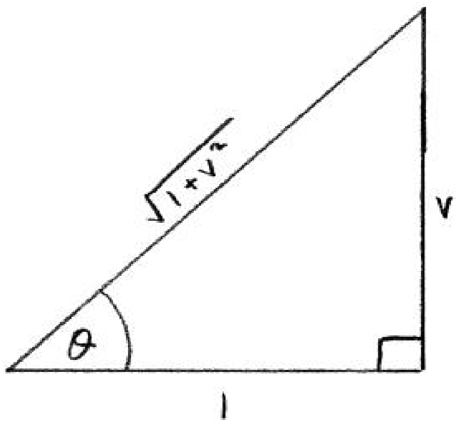
e. **METHOD 1**

$\frac{\pi}{4} - t = \arctan v$ ***M1***

$t = \frac{\pi}{4} - \arctan v$

$s = \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{\pi}{4} + \arctan v\right)\right]$

$s = \ln\left[\sqrt{2}\cos(\arctan v)\right]$ ***M1A1***



$s = \ln\left[\sqrt{2}\cos\left(\arccos\frac{1}{\sqrt{1+v^2}}\right)\right]$ ***A1***

$= \ln\frac{\sqrt{2}}{\sqrt{1+v^2}}$

$= \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

METHOD 2

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$

$= -\ln\sec\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + \tan^2\left(\frac{\pi}{4} - t\right)} - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + v^2} - \ln\cos\frac{\pi}{4}$ ***A1***

$= \ln\frac{1}{\sqrt{1+v^2}} + \ln\sqrt{2}$ ***A1***

$= \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

METHOD 3

$v\frac{dv}{ds} = -v^2 - 1$ ***M1***

$\int\frac{v}{v^2+1}dv = -\int 1ds$ ***M1***

$\frac{1}{2}\ln(v^2 + 1) = -s + k$ ***A1***

when $s = 0\,,\ t = 0 \Rightarrow v = 1$

$\Rightarrow k = \frac{1}{2}\ln 2$ ***A1***

$\Rightarrow s = \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

[4 marks]

Examiners report

a. This proved to be the most challenging question in section B with only a very small number of candidates producing fully correct answers. Many candidates did not realise that part (a) was a differential equation that needed to be solved using a method of separating the variables. Without this, further progress with the question was difficult. For those who did succeed in part (a), parts (b) and (c) were relatively well done. For the minority of candidates who attempted parts (d) and (e) only the best recognised the correct methods.

b. This proved to be the most challenging question in section B with only a very small number of candidates producing fully correct answers. Many candidates did not realise that part (a) was a differential equation that needed to be solved using a method of separating the variables. Without this, further progress with the question was difficult. For those who did succeed in part (a), parts (b) and (c) were relatively well done. For the minority of candidates who attempted parts (d) and (e) only the best recognised the correct methods.

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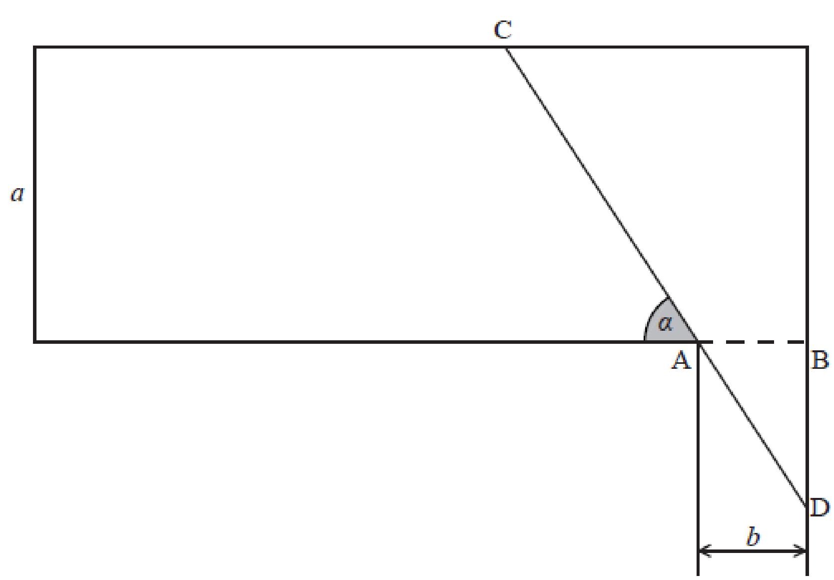
The diagram shows the plan of an art gallery a metres wide. [AB] represents a doorway, leading to an exit corridor b metres wide. In order to remove a painting from the art gallery, CD (denoted by L) is measured for various values of α , as represented in the diagram.



a. If α is the angle between [CD] and the wall, show that $L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$, $0 < \alpha < \frac{\pi}{2}$. [3]

b. If $a = 5$ and $b = 1$, find the maximum length of a painting that can be removed through this doorway. [4]

c. Let $a = 3k$ and $b = k$. [3]

Find $\frac{dL}{d\alpha}$.

d. Let $a = 3k$ and $b = k$. [6]

Find, in terms of k, the maximum length of a painting that can be removed from the gallery through this doorway.

e. Let $a = 3k$ and $b = k$. [2]

Find the minimum value of k if a painting 8 metres long is to be removed through this doorway.

Markscheme

a. $L = \text{CA} + \text{AD}$ ***M1***

$\sin\alpha = \frac{a}{\text{CA}} \Rightarrow \text{CA} = \frac{a}{\sin\alpha}$ ***A1***

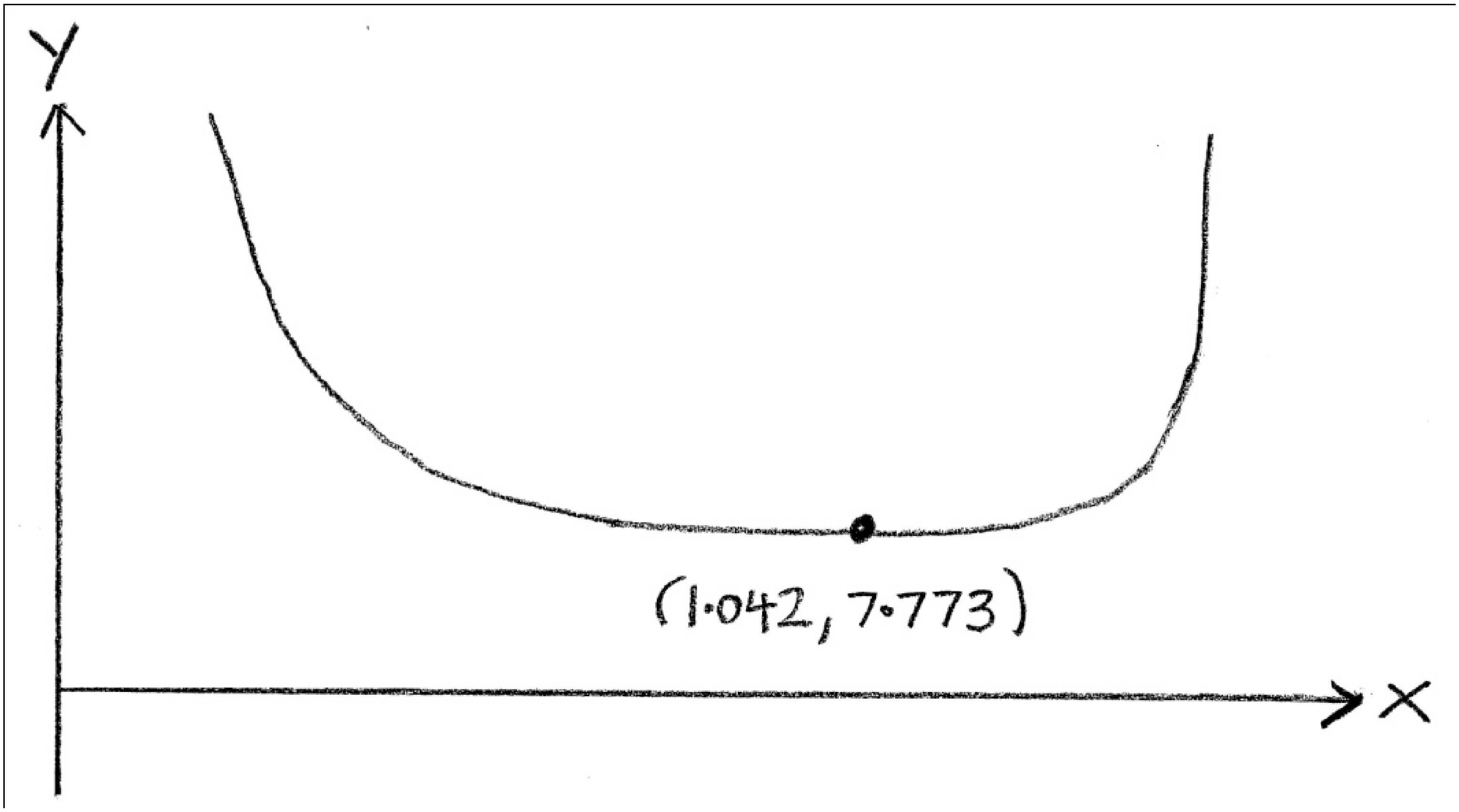
$\cos\alpha = \frac{b}{\text{AD}} \Rightarrow \text{AD} = \frac{b}{\cos\alpha}$ ***A1***

$L = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$ ***AG***

[2 marks]

b. $a = 5$ and $b = 1 \Rightarrow L = \frac{5}{\sin\alpha} + \frac{1}{\cos\alpha}$

METHOD 1



(M1)

minimum from graph $\Rightarrow L = 7.77$ ***(M1)A1***

minimum of L gives the max length of the painting ***R1***

[4 marks]

METHOD 2

$\frac{dL}{d\alpha} = \frac{-5\cos\alpha}{\sin^2\alpha} + \frac{\sin\alpha}{\cos^2\alpha}$ ***(M1)***

$\frac{dL}{d\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = 5 \Rightarrow \tan\alpha = \sqrt[3]{5}$ $(\alpha = 1.0416...)$ ***(M1)***

minimum of L gives the max length of the painting ***R1*** maximum length = 7.77 ***A1***

[4 marks]

c. $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{-3k\cos\alpha}{\sin^2\alpha} + \frac{k\sin\alpha}{\cos^2\alpha}$ (or equivalent) ***M1A1A1***

[3 marks]

d. $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{-3k\cos^3\alpha + k\sin^3\alpha}{\sin^2\alpha\cos^2\alpha}$ ***(A1)***

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \frac{\sin^3\alpha}{\cos^3\alpha} = \frac{3k}{k} \Rightarrow \tan\alpha = \sqrt[3]{3} \quad (\alpha = 0.96454\ldots)$ ***M1A1***

$\tan\alpha = \sqrt[3]{3} \Rightarrow \frac{1}{\cos\alpha} = \sqrt{1+\sqrt[3]{9}} \quad (1.755\ldots)$ ***(A1)***

and $\frac{1}{\sin\alpha} = \frac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}} \quad (1.216\ldots)$ ***(A1)***

$L = 3k\left(\frac{\sqrt{1+\sqrt[3]{9}}}{\sqrt[3]{3}}\right) + k\sqrt{1+\sqrt[3]{9}} \quad (L = 5.405598\ldots k)$ ***A1 N4***

[6 marks]

e. $L \leqslant 8 \Rightarrow k \geqslant 1.48$ ***M1A1***

the minimum value is 1.48

[2 marks]

Examiners report

a. Part (a) was very well done by most candidates. Parts (b), (c) and (d) required a subtle balance between abstraction, differentiation skills and use of GDC.

In part (b), although candidates were asked to justify their reasoning, very few candidates offered an explanation for the maximum. Therefore most candidates did not earn the R1 mark in part (b). Also not as many candidates as anticipated used a graphical approach, preferring to use the calculus with varying degrees of success. In part (c), some candidates calculated the derivatives of inverse trigonometric functions. Some candidates had difficulty with parts (d) and (e). In part (d), some candidates erroneously used their alpha value from part (b). In part (d) many candidates used GDC to calculate decimal values for α and L. The premature rounding of decimals led sometimes to inaccurate results. Nevertheless many candidates got excellent results in this question.

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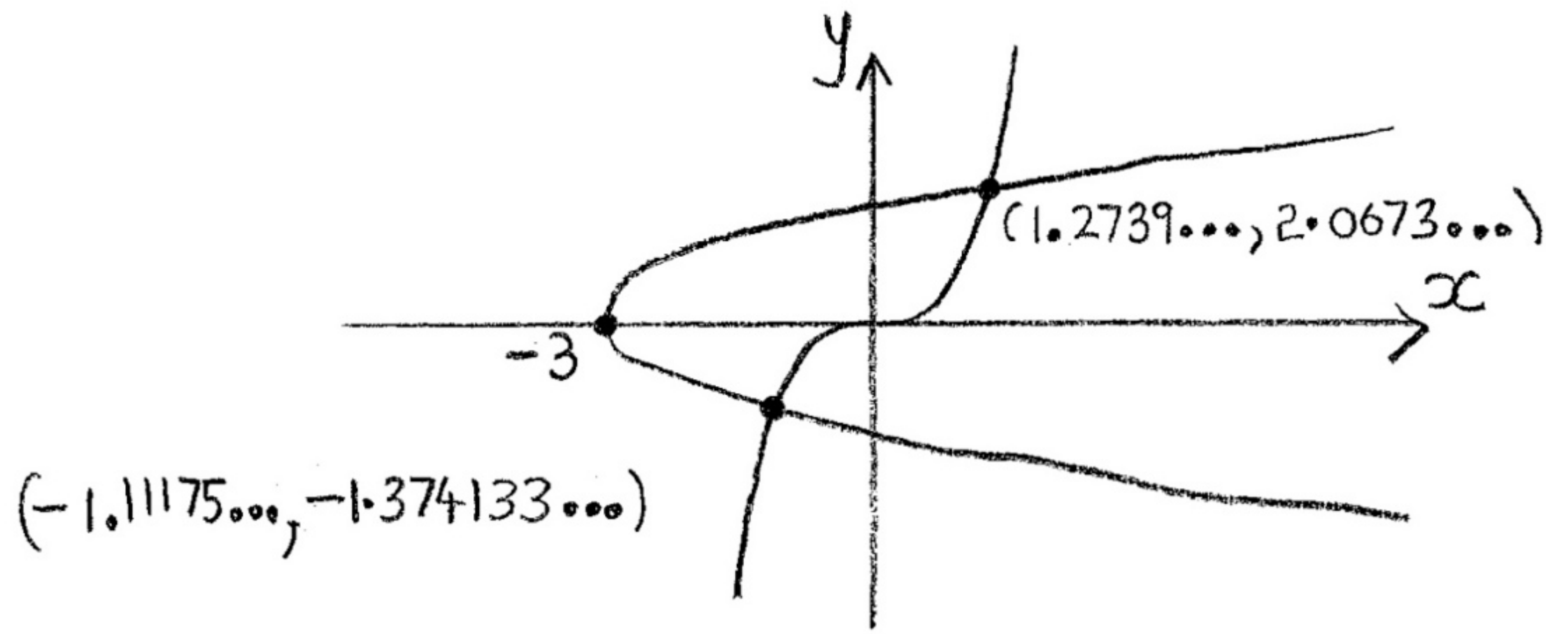
Find the area of the region enclosed by the curves $y = x^3$ and $x = y^2 - 3$.

Markscheme



intersection points ***A1A1***
Note: Only either the *x*-coordinate or the *y*-coordinate is needed.

EITHER
$x = y^2 - 3 \Rightarrow y = \pm\sqrt{x+3} \quad \left(\text{accept } y = \sqrt{x+3}\right)$ ***(M1)***
$A = \int_{-3}^{-1.111...} 2\sqrt{x+3}\,\mathrm{d}x + \int_{-1.111...}^{1.2739...} \sqrt{x+3} - x^3 \mathrm{d}x$ ***(M1)A1A1***
= 3.4595... + 3.8841...
= 7.34 (3sf) ***A1***
OR
$y = x^3 \Rightarrow x = \sqrt[3]{y}$ ***(M1)***
$A = \int_{-1.374...}^{2.067...} \sqrt[3]{y} - (y^2 - 3)\mathrm{d}y$ ***(M1)A1***
= 7.34 (3sf) ***A2***
[7 marks]

Examiners report

This question proved challenging to most candidates. Just a few candidates were able to calculate the exact area between curves. Those candidates who tried to express the functions in terms *x* of instead of *y* showed better performances. Determining only $\sqrt{x+3}$ was a common error and forming appropriate definite integrals above and below the *x*-axis proved difficult. Although many candidates attempted to sketch the graphs, many found only one branch of the parabola and only one point of intersection; as the graph of the parabola was not complete, many candidates did not know which area they were trying to find. Not many split the integral correctly to find areas that would add up to the result. Premature rounding was usually seen and consequently final answers proved inaccurate.

Consider the graphs $y = \mathrm{e}^{-x}$ and $y = \mathrm{e}^{-x} \sin 4x$, for $0 \leqslant x \leqslant \frac{5\pi}{4}$.

(a) On the same set of axes draw, on graph paper, the graphs, for $0 \leqslant x \leqslant \frac{5\pi}{4}$. Use a scale of 1 cm to $\frac{\pi}{8}$ on your x-axis and 5 cm to 1 unit on your y-axis.

(b) Show that the x-intercepts of the graph $y = \mathrm{e}^{-x}$ sin 4x are $\frac{n\pi}{4}$, $n = 0, 1, 2, 3, 4, 5$.

(c) Find the x-coordinates of the points at which the graph of $y = \mathrm{e}^{-x} \sin 4x$ meets the graph of $y = \mathrm{e}^{-x}$. Give your answers in terms of π.

(d) (i) Show that when the graph of $y = \mathrm{e}^{-x} \sin 4x$ meets the graph of $y = \mathrm{e}^{-x}$, their gradients are equal.

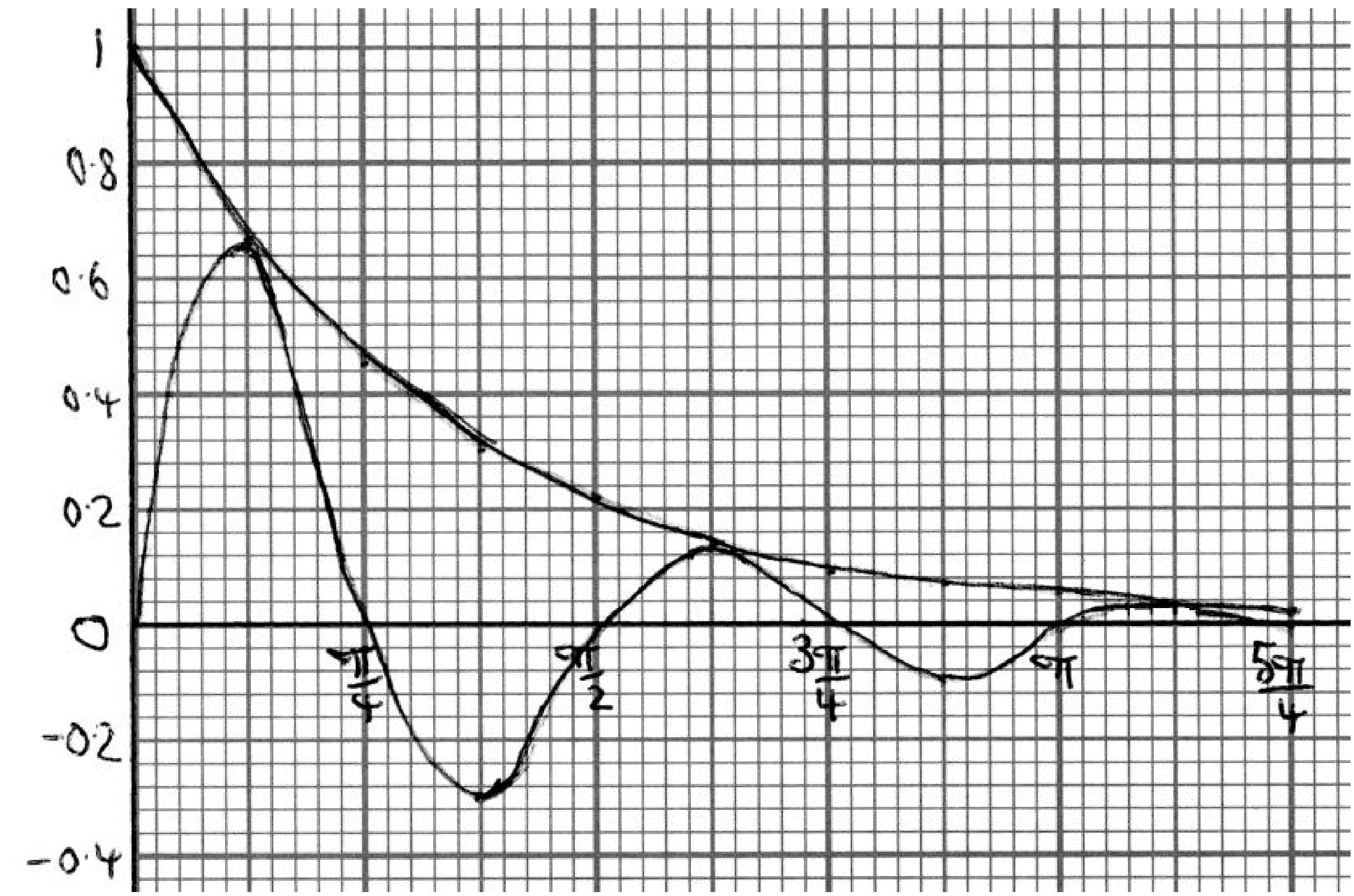
(ii) Hence explain why these three meeting points are not local maxima of the graph $y = \mathrm{e}^{-x} \sin 4x$.

(e) (i) Determine the y-coordinates, y_1 , y_2 and y_3, where $y_1 > y_2 > y_3$, of the local maxima of $y = \mathrm{e}^{-x} \sin 4x$ for $0 \leqslant x \leqslant \frac{5\pi}{4}$. You do not need to show that they are maximum values, but the values should be simplified.

(ii) Show that y_1 , y_2 and y_3 form a geometric sequence and determine the common ratio r .

Markscheme

(a)



A3

Note: Award ***A1*** for each correct shape,

A1 for correct relative position.

[3 marks]

(b) $\mathrm{e}^{-x}\sin(4x) = 0$ ***(M1)***

$\sin(4x) = 0$ ***A1***

$4x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$ ***A1***

$x = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}$, \(\frac{5\pi }{4}\) ***AG***

[3 marks]

(c) $\mathrm{e}^{-x} = \mathrm{e}^{-x}\sin(4x) = 0$ or reference to graph

$\sin 4x = 1,$ ***M1***

$\sin 4x = 1, frac\pi 2, frac5\pi 2, frac9\pi 2$ ***A1***

$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ ***A1*** ***N3***

[3 marks]

(d) (i) $y = \mathrm{e}^{-x}\sin 4x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}\sin 4x + 4\mathrm{e}^{-x}\cos 4x$ ***M1A1***

$y = \mathrm{e}^{-x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}$ ***A1***

verifying equality of gradients at one point ***R1***

verifying at the other two ***R1***

(ii) since $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ at these points they cannot be local maxima ***R1***

[6 marks]

(e) (i) maximum when $y' = 4\mathrm{e}^{-x} \cos 4x - \mathrm{e}^{-x} \sin 4x = 0$ ***M1***

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+\pi}{4}, \frac{\arctan(4)+2\pi}{4}, \ldots$

maxima occur at

$x = \frac{\arctan(4)}{4}, \frac{\arctan(4)+2\pi}{4}, \frac{\arctan(4)+4\pi}{4}$ ***A1***

so $y_1 = \mathrm{e}^{-\frac{1}{4}(\arctan(4))} \sin(\arctan(4)) \quad (= 0.696)$ ***A1***

$y_2 = \mathrm{e}^{-\frac{1}{4}(\arctan(4)+2\pi)} \sin(\arctan(4) + 2\pi)$ ***A1***

$\left(= \mathrm{e}^{-\frac{1}{4}(\arctan(4)+2\pi)} \sin(\arctan(4)) = 0.145\right)$

$y_3 = \mathrm{e}^{-\frac{1}{4}(\arctan(4)+4\pi)} \sin(\arctan(4) + 4\pi)$ ***A1***

$\left(= \mathrm{e}^{-\frac{1}{4}(\arctan(4)+4\pi)} \sin(\arctan(4)) = 0.0301\right)$ ***N3***

(ii) for finding and comparing $\frac{y_3}{y_2}$ and $\frac{y_2}{y_1}$ ***M1***

$r = \mathrm{e}^{-\frac{\pi}{2}}$ ***A1***

Note: Exact values must be used to gain the ***M1*** and the ***A1***.

[7 marks]

Total [22 marks]

Examiners report

Although the final question on the paper it had parts accessible even to the weakest candidates. The vast majority of candidates earned marks on part (a), although some graphs were rather scruffy. Many candidates also tackled parts (b), (c) and (d). In part (b), however, as the answer was given, it should have been clear that some working was required rather than reference to a graph, which often had no scale indicated. In part d(i), although the functions were usually differentiated correctly, it was often the case that only one point was checked for the equality of the gradients. In part e(i) many candidates who got this far were able to determine the y-coordinates of the local maxima numerically using a GDC, and that was given credit. Only the exact values, however, could be used in part e(ii).